The interdependence of bank capital and liquidity*

Elena Carletti Itay Goldstein Agnese Leonello

This Version: January 2020

Abstract

This paper analyzes the role of liquidity regulation and its interaction with capital requirements. We first introduce costly capital in a bank run model with endogenous bank portfolio choice and run probability, and show that capital regulation is the only way to restore the efficient allocation. We then enrich the model to include fire sales, and show that capital and liquidity regulation are complements. The key implications of our analysis are that the optimal regulatory mix should be designed considering both sides of banks' balance sheet, and that its effectiveness depend on the costs of both capital and liquidity.

Keywords: illiquidity, insolvency, fire sales, optimal regulation

JEL classifications: G01, G21, G28

*Correponding author: Itay Goldstein. Authors' contact details: I. Goldstein: Wharton School, University of Pennsylvania, 2321 SH/DH, 3620 Locust Walk Philadelphia, PA 19104-6367; phone: +1 215-746-0499; email: itayg@wharton.upenn.edu; E. Carletti: Bocconi University, IGIER and CEPR, Department of Finance, Via Roentgen, 1, 20136 Milan (Italy); phone: +39 02 5836 5953; email: elena.carletti@unibocconi.it; A. Leonello: European Central Bank, Sonnemannstraße 20, 60314 Frankfurt am Main (Germany), phone: +496913447327, email: agnese.leonello@ecb.europa.eu. We are grateful to participants at the 2019 FIRS, the 2019 AFA, the 2018 NBER summer workshop "Risk of Financial Institutions", the 2018 Oxford Financial Intermediation Theory (OxFIT) conference, the conference of the ESCB 3rd Reseach Cluster on Financial Stability, Macroprudential Regulation and Microprudential Supervision, the 1st Annual Conference of the JRC Community of Practice in Financial Research "The Resilience of the Financial System" (CoPFiR), the "Lisbon Meetings in Game Theory and Applications", the ECB Fourth Macroprudential Policy and Research Conference; and to seminar participants at the University of Cambridge, the University College Dublin, Bank of Italy, the European Central Bank, the Frankfurt School of Finance and Management, the Bundesbank, the University of Pireus, the University of Rome "Tor Vergata", the University of Naples "Federico II", Cambridge University and the University College Dublin. We also wish to thank for useful comments and suggestions Toni Anhert, Nicolas Inostroza, Philipp Koenig, Ye Li, Xuewen Liu, Anatoli Segura, Hyun Shin, Anjan Thakor, Jean Tirole and Xavier Vives. We also thank Marcello Parravani for excellent research assistance. Elena Carletti acknowledges financial support from Baffi Carefin Center at Bocconi University. The views expressed here are the authors' and do not reflect those of the ECB or the Eurosystem.

1 Introduction

The 2007-2009 financial crisis was a milestone for financial regulation, leading to significant reforms to the existing capital regulation and the introduction of a new set of liquidity requirements. In particular, banks have been required to hold higher capital buffers to reduce their exposure to solvency-driven crises and, at the same time, to increase their liquidity holdings to reduce liquidity mismatch and the consequent risk of liquidity-driven crises. The introduction of a new set of liquidity requirements, namely the Liquidity Coverage Ratio (LCR) and the Net Stable Funding Ratio (NSFR), as complements to the existing and improved capital-based regulation, has led to a debate in the academic and policy arena on the effective need of all these regulatory tools, their interaction, as well as their potential contrasting effects for financial stability and welfare.

Bank (il)liquidity and (in)solvency are closely intertwined concepts and often difficult to tell apart when a crisis manifests (see e.g., Goodhart, 1999). On the one hand, liquidity-driven crises can spur solvency issues; on the other hand, fears about bank solvency may precipitate liquidity problems. Furthermore, when a crisis is underway and a bank faces a large outflow of funds, it becomes very difficult to assess the ultimate source of these withdrawals, which, in turn, may limit policymakers' ability to intervene effectively.

It is precisely this close link between solvency and liquidity crises that motivates the discussion about the joint effects that capital and liquidity regulation have on financial stability, and ultimately, welfare. In particular, what are the effects of changes in the level of bank capitalization and portfolio liquidity on bank stability? How do capital and liquidity requirements interact in affecting the probability of bank failure? Are capital and liquidity requirements equally effective in curbing fragility and/or fostering productive investments?

To tackle these questions, we start by building a simple model, where we introduce capital in a two-period global games model with endogenous portfolio choice. This allows us to derive a simple working framework, where the run probability is endogenous to banks' balance sheet choices and capital is costly so that there is scope for capital regulation. The model has one bank issuing short term debt and equity, and investing in a risky portfolio consisting of liquid and illiquid assets, whose final return increases with the fundamentals of the economy. The portfolio composition determines the trade-off between intermediate and final date portfolio returns, whereby a higher proportion of liquid assets leads to a higher (safe) return at the interim date, but to a lower (risky) return at the final date.

The portfolio liquidity and the capital structure affect debt holders' decision to roll over their debt and, consequently, the likelihood of a bank failure. In this setting, the bank fails as a consequence of a massive withdrawal of funds by debt holders at the interim date (i.e., a run). A debt holder's withdrawal decision is based on an imperfect signal regarding the date 2 bank portfolio return that each debt holder receives at date 1, as the signal provides information about the fundamental of the economy and the actions of the other debt holders.

As standard in the global games literature (see e.g., Morris and Shin, 1998, 2003; Rochet and Vives, 2004; Goldstein and Pauzner, 2005), the equilibrium outcome is that a run occurs when the fundamentals of the economy are below a unique threshold. Within the range of fundamentals where they occur, crises can be classified into either solvency- or liquidity-driven crises. The former happen at the lower part of the crisis region where the signal on the fundamentals is so low that not rolling over the debt claim at the interim date is a dominant strategy for debt holders. The latter occur for an intermediate region of fundamentals and are due to the presence of strategic complementarity among debt holders, in that each of them does not roll over out the self-fulfilling belief that others will do the same.

We show the crisis threshold crucially depends on the level of bank capitalization and its portfolio liquidity. In particular, higher capitalization or increased portfolio liquidity increase the crisis probability for a bank with very little capital and/or very illiquid portfolios, while decreasing it for a bank with intermediate levels of capital or liquidity. Finally, for a bank with high initial levels of capital or liquidity, higher capitalization is beneficial for stability, while more portfolio liquidity is detrimental. These results hinge on the fact that capital and liquidity affect debt holders' payoff at both date 1 and date 2, and debt holders value the payoffs at either date differently based on the initial bank balance sheet structure.

This comparative statics exercise delivers some initial implications for the design of regulation. In particular, it hints to the fact that capital and liquidity requirements should be designed considering both sides of banks' balance sheets. In this respect, our analysis supports regulatory instruments like the risk-weighted capital ratio, the liquidity coverage ratio and the net stable funding ratio that essentially specify a ratio between banks' assets and liabilities, in line also with Cecchetti and Kashyap (2018). Furthermore, small changes in the level of capitalization and/or portfolio liquidity may have undesirable consequences for some banks, especially for those who would need to strengthen their stability the most, thus highlighting the importance to calibrate the size of the regulatory intervention to the bank's specific balance sheet conditions and risk exposures.

Building on the comparative statics exercise, we then analyze the bank's choice of capital structure, portfolio liquidity and debt holders' repayment in the unregulated equilibrium and show that the allocation is inefficient. The bank chooses a level of capitalization that is too low, thus exposing itself excessively to runs and ultimately foregoing the return of too many good investment projects. This inefficiency hinges on the existence of a wedge between the private and the social cost of capital, which prevents the bank from choosing the level of capital that would eliminate inefficient crises.

This inefficiency leaves scope for regulation. We show that capital requirements are the only effective tool in restoring the efficient allocation, where inefficient crises can be eliminated and the optimal portfolio allocation between liquid and illiquid assets is achieved. Once capital is determined by regulation, the bank chooses the socially optimal level of liquidity as there is no wedge between the private and the social cost of liquidity.

In sum, our baseline framework pictures an economy whose inefficiency hinges on the bank's preference for debt financing over equity, and it is consistent with the pre-crisis view of the central role of capital requirements in the regulatory framework. Thus, our baseline framework offers an ideal framework where to embed additional frictions justifying the need of liquidity regulation and it allows us to study the interaction between liquidity and capital requirements.

We next enrich the model to include the possibility of fire sales. Specifically, we consider the presence of multiple symmetric banks sharing the same fundamentals. The key difference relative to the baseline model is that banks sell shares of their portfolios to meet early withdrawals in a secondary asset market to outside investors. Outside investors are endowed with limited resources and may be less able than banks in managing the portfolios they acquire. As a result, the (per unit) amount that a bank can raise from the market does no longer correspond only to their portfolio liquidity choice, but it also depends on aggregate market conditions. Transferring assets outside the banking sector then may entail a loss of resources, which increases with the size and illiquidity of the pool of assets on sale in the market.

This specification modifies the analysis in two important dimensions. First, it affects debt holders' rollover decision by introducing another source of strategic complementarity. This implies that banks are strategically connected and their failures may spur from contagion: Banks fail because their debt holders are concerned about the health of other banks in the system. Second, the extended framework features an additional inefficiency in banks' decisions since banks do not internalize the effect that their individual choices have on the secondary asset market and consequently on the other banks.

We first show that in the economy with multiple banks, crises are more likely and also more costly in that good projects are liquidated more often and their premature liquidation entails a larger cost than in the baseline model. We then show that in this scenario, both liquidity and capital requirements are needed, since the economy now features a wedge between the private and the social cost of liquidity in addition to the one for capital. As a result, liquidity and capital requirements are now complements: The former are used to prevent the occurrence of fire sales; the latter are needed to prevent the premature liquidation of profitable investment projects. As in the baseline model, the regulatory allocation depends on the cost of capital and liquidity. Specifically, the equilibrium features inefficient crises and/or fire sales when the cost of capital and/or liquidity is high, while both inefficiencies are eliminated through an appropriate combination of capital and liquidity requirements otherwise.

A number of recent papers have looked at the role and implications of the newly introduced liquidity regulation, also in connection with capital requirements (see, e.g., Walther, 2015; Calomiris, Heider and Hoerova, 2015; Diamond and Kashyap, 2016; Macedo and Vicente, 2017). Closer to us are other papers using the global-games methodology to study the implications of capital and liquidity on the probability of banking crises: Vives (2014), König (2015), and Schilling (2016). However, differently from these papers, we endogenize bank capital structure, portfolio liquidity and debt holders' returns. This allows us to highlight the inefficiencies of the market equilibrium and study the optimal regulation. The only other paper that attempts to do this, as far as we know, is Kashyap, Tsomocos and Vardoulakis (2019). However, our framework differs from theirs in terms of sources of inefficiencies and uncertainty driving debt holders' withdrawal decisions. Concerning the former, Kashyap et al. (2019) focus on the interaction between run and credit risk, while our paper looks at the interaction between run risk, liability structure and fire sales externalities. Concerning the latter, in their framework, the uncertainty is on the liquidation value of bank loans, while in ours, it pertains to the bank long-term project return. Overall, these differences lead to different debt holders' payoff structure and a different interaction between capital and liquidity regulation. Most importantly, our framework is much simpler than theirs and lends itself more easily to analytical solutions. We think this is an advantage for the future development of the literature and the ability to embed such models in actual policymaking.

The key aspect of our study is the ability to endogenously derive the probability of a banking crisis and study how it is affected by changes in bank capitalization and portfolio liquidity. To do this, we rely on the global game techniques as developed in the literature originating with Carlsson and van Damme (1993) (see Morris and Shin, 2003 for a survey on the theory and applications of global games). Our paper is close to three contributions in this literature. First, it shares the idea of rollover game with Eisenbach (2017), although in a framework where banks also raise equity and choose the liquidity-return trade-off of their portfolio. Second, it faces the same technical challenge of characterizing the existence of a unique equilibrium in the absence of global strategic complementarities as in Goldstein and Pauzner (2005). Finally, the extended framework deals with the characterization of a unique crisis threshold in the presence of two types of strategic complementarities- within and between banks- and multiple banks, thus extending the analysis of Goldstein (2005).

The paper proceeds as follows. Section 2 presents the baseline model. Section 3 characterizes the equilibrium with one bank, while Section 4 that with multiple banks. In both sections, we first derive debt holders' rollover decision. Then, we characterize banks' choices and identify the inefficiencies of the unregulated equilibrium. Finally, we analyze the effectiveness of regulation in addressing them. Section 5 contains concluding remarks. All proofs are contained in the Online Appendix.

2 The baseline model with one bank

We start by considering a simple three date ($t = 0, 1, 2$) economy, with a representative bank and a continuum $[0, 1]$ of risk neutral investors, each endowed with one unit of resources at date 0 and nothing thereafter. At date 0 the bank raises one unit of funds from investors and invests in a risky portfolio. Below we specify both the funding and the assets of the bank. We then discuss the key assumptions of the baseline model.

2.1 The framework

At date 0, the bank determines the composition of its portfolio between liquid and illiquid assets. In particular, it chooses the level of portfolio liquidity $\ell \in [0, 1]$. For each unit invested at date 0, the portfolio yields a per unit return ℓ at date 1 and $R(\theta)(1 - \alpha\ell)$ at date 2, with $\alpha > 0$, $\theta \sim U[0, 1]$ representing the aggregate state of the economy, $R'(\theta) > 0$, $R(0) = 0$ and $E_\theta[R(\theta)(1 - \alpha\ell)] > 1$. The specification entails a standard liquidity-return trade-off: A more liquid portfolio (i.e., with a higher ℓ) yields a higher date 1 return, but a lower expected return at date 2, so that through ℓ the bank chooses a point on a liquidity-return frontier. In other words, the parameter α captures the cost of holding a liquid portfolio in terms of lower date 2 return.

On the funding side, the bank raises a fraction k of funds as equity and the remaining fraction $1-k$ as short-term debt. Equity finance entails a cost $\rho > 1$ for the bank, representing the opportunity cost of funds for equity holders. By contrast, the debt contract specifies a promised (gross) interest rate $r_1 = 1$ to debt holders withdrawing (or, equivalently, not rolling over) at date 1 and $r_2 \geq r_1$ date 2. The debt market is perfectly competitive so the bank will always set r_2 at the level required for debt holders to recover their opportunity cost of funds, which we normalize to 1, and thus be willing to participate.

The aggregate state of the economy θ is realized at the beginning of date 1, but is not publicly observed until date 2. After θ is realized, at date 1 each debt holder receives a private signal s_i of the form

$$s_i = \theta + \varepsilon_i, \tag{1}$$

where ε_i are small error terms, which are independently and uniformly distributed over $[-\varepsilon, +\varepsilon]$. Based on this signal, debt holders decide whether to withdraw at date 1 or roll over the debt until date 2.

The bank satisfies early withdrawals by liquidating a share of its portfolio. The bank fails at either date when it does not have enough resources to repay the promised due debt repayments. When this occurs, debt holders receive a share of the bank's available resources while equity holders obtain nothing.

The timing of the model is as follows. At date 0 the bank chooses the capital structure $\{k, 1-k\}$, the debt repayment r_2 and the level of portfolio liquidity ℓ so to maximize its expected profit. At date 1, after receiving the private signal about the state of the fundamentals θ, each debt holder decides whether to roll the debt over. At date 2, the bank portfolio return realizes and all claims are paid, if the bank is solvent. The model is solved backwards.

2.2 Discussion of the assumptions

Portfolio specification. Our portfolio specification can be seen as a reduced form of a more standard approach whereby the bank chooses to allocate a fraction x in liquid assets returning $0 < T \leq 1$ at either date and $1-x$ in illiquid and risky assets yielding $\lambda < T$ at date 1 and $R(\theta)$ at date 2. Thus, we can specify the return of the portfolio at $t = 1$ as

$$\ell = xT + (1-x)\lambda, \tag{2}$$

and that at $t = 2$ as $xT + (1-x) R(\theta)$. These portfolio returns are equivalent to those in our framework with α being equal to

$$\alpha(\theta, \ell) = \left[1 - \frac{T}{R(\theta)}\right] \frac{\ell - \lambda}{\ell (T - \lambda)}.$$

This particular value of α equates the date 2 portfolio returns in the two frameworks, i.e.,

$$xT + R(\theta)(1-x) \equiv R(\theta)(1 - \alpha\ell), \tag{3}$$

once we substitute the expression for $x = \frac{\ell - \lambda}{T - \lambda}$ from (2) into (3).

Specifying a value for α, which depends on θ and ℓ, enriches the framework without affecting our qualitative results regarding the existence of potentially non-monotone effects of capital and liquidity on the run thresholds. Yet, our reduced form with a generic α is more convenient in terms of tractability and allows us to focus on portfolio early liquidation rather than on single assets.

Short term debt. Our framework takes the optimality of short-term debt as given. This has been justified in the literature in the presence of asymmetric information problems in credit markets (see, e.g., Flannery, 1986; and Diamond, 1991), conflicts between banks' managers and shareholders (see e.g., Calomiris and Kahn, 1991; Diamond and Rajan, 2001; and Eisenbach, 2017) and idiosyncratic liquidity shocks to bank depositors (e.g., Diamond and Dybvig, 1983).

The latter explanation could be easily adopted in the context of our model. By assuming risk averse debt holders with early liquidity needs, the result $r_1 > 1$ would emerge optimally as in Goldstein and Pauzner (2005). However, this would complicate the analysis without bringing any new insights in terms of the role of bank capital and liquidity for run risk and investors' coordination problem. Thus, for tractability, we simply assume risk neutral agents and abstract from investors' liquidity needs. As a consequence, we take r_1 to be given, and for simplicity, to be equal to 1.

Cost of funding: The assumption $\rho > 1$ implies that bank capital is a more expensive form of financing than debt. This assumption, which is standard in the literature (see e.g., Hellmann, Murdock and Stiglitz, 2000; Repullo, 2004; Allen, Carletti and Marquez, 2011), has been recently endogenized on the basis of market segmentation (see, e.g., Allen, Carletti and Marquez, 2015; and Carletti, Marquez and Petriconi, 2019), the presence of implicit or explicit government guarantees to bank debt (see, e.g., Admati and Hellwig, 2013) or the existence of costs associated with the issuance of outside equity (see Harris, Opp and Opp, 2017) and empirically validated based on a different tax treatment between equity and debt (see e.g., Schepens, 2016).[1] Following Allen et al. (2015), we could endogenize ρ as the unit price of capital in a market with a fixed supply of capital K from investors and a demand as coming from the bank's optimal capital structure. As long as K is not too large, $\rho > 1$ would emerge in equilibrium with the precise value of ρ being an increasing function of the bank's capital k. This would imply the same trade-off in the bank's capital structure decision between higher cost of funding and run risk as in our baseline model, with the difference that the bank would have an extra incentive to economize on capital in order to reduce its cost. Assuming $\rho > 1$ eliminates this latter effect without affecting the key insights of our analysis on the interaction between capital and liquidity regulation.

[1] In most jurisdictions, the cost of debt is tax-deductible, while dividends are not. Schepens (2016) shows that a reduction in the tax discrimination between debt and equity financing leads to a significant increase in bank capital ratios.

3 The equilibrium with one bank

In this section, we derive the unregulated equilibrium of our basic framework with one bank and characterize the regulatory intervention. We first characterize debt holders' rollover decisions at date 1 for given levels of capital k, portfolio liquidity ℓ and debt repayment r_2. This allows us to pin down the probability of a bank failure. Then, we characterize the bank's choice of capital, portfolio liquidity and debt repayment at date 0. Finally, we analyze the regulation needed to remove the inefficiencies of the unregulated equilibrium.

3.1 Debt holders' rollover decisions

Each debt holder decides whether to withdraw at date 1 based on the signal s_i he receives since this provides information on both θ and other debt holders' actions. When the signal is high, a debt holder attributes a high posterior probability to the event that the bank portfolio yields a high return and, at the same time, he infers that the other debt holders have also received a high signal. This overall lowers his belief about the likelihood of a bank failure and, as a result, also his own incentive to withdraw at date 1. Conversely, when the signal is low, a debt holder has a high incentive not to roll over the debt, as he attributes a high likelihood to the possibility that the return of the bank portfolio is low and that the other investors withdraw their debt claim at date 1. This suggests that debt holders withdraw at date 1 when the signal is low enough, and roll their debt claims over until date 2 when the signal is sufficiently high.

To show this formally, we first examine two regions corresponding to extremely bad and extremely good realization of the aggregate state variable θ, where each debt holder's action is based only on the realization of θ irrespective of his beliefs about the others' behavior. We start with the lower region.

Lower dominance region. When θ is very low, not rolling over the debt at date 1 is a dominant strategy for every debt holder. This is the case when, upon receiving his signal, a debt holder expects to receive at date 2 a lower payoff than the return 1 he would obtain by withdrawing at date 1, even if all other debt holders wait until date 2. Since $r_2 \geq 1$, this occurs when the bank fails at date 2 and the debt holder obtains the pro-rata share $\frac{R(\theta)(1-\alpha\ell)}{(1-k)}$. We then denote as $\underline{\theta}(k,\ell)$ the value of θ that solves

$$\frac{R(\theta)(1-\alpha\ell)}{(1-k)} = 1 \tag{4}$$

so that the interval $[0, \underline{\theta}(k,\ell))$ identifies the range of values of θ where banking crises are only driven by low fundamentals.[2]

Upper dominance region. The upper dominance region of θ corresponds to the range $[\overline{\theta}, 1]$ in which the state of the economy is so good that rolling over is a dominant strategy. As in Goldstein and Pauzner (2005), we construct this region by assuming that in the range $[\overline{\theta}, 1]$ the bank investment is safe, i.e., it yields $R(1)(1-\alpha\ell) > 1$ both at dates 1 and 2. Given that $1 \leq r_2 < R(1)(1-\alpha\ell)$, this ensures that repaying 1 to the withdrawing debt holders does not affect the bank's ability to repay r_2 to the debt holders rolling over the debt until date 2. Then, when an investor receives a signal such that he believes that the fundamental θ is in the upper dominance region, he is certain to receive the promised payment r_2, irrespective of his beliefs on other debt holders' actions. As a consequence, he does not have any incentive to withdraw early. In what follows, we assume that $\overline{\theta} \to 1$.

The Intermediate Region. When the signal indicates that θ is in the intermediate range $[\underline{\theta}(k,\ell), \overline{\theta})$, a debt holder's rollover decision depends on the realization of θ as well as on his beliefs regarding other debt holders' actions. Debt holders may have the incentive not to rollover their debt claims at the interim date as they fear that others would do the same. Their concern is that a large number of withdrawals at date 1 would force a massive liquidation of the bank portfolio, thus depleting bank's available resources at date 2 and, in turn, their expected payoff. Denoting as n the proportion of debt holders withdrawing at date 1, the bank liquidates $\frac{(1-k)n}{\ell}$ units of portfolio, so that a debt holder's payoff differential between rolling over until date 2

[2] For the lower dominance region to exist, it must be the case that there are feasible values of θ for which all debt holders receive a signal below $\underline{\theta}(k,\ell)$. Since the noise contained in the signal is at most ε, when $s_i < \underline{\theta}(k,\ell) - \varepsilon$ all debt holders receive a signal below $\underline{\theta}(k,\ell)$. This holds when $\theta < \underline{\theta}(k,\ell) - 2\varepsilon$.

and withdrawing at date 1 is given by

$$v(\theta,n)=\begin{cases} r_2-1 & \text{if } 0\leq n<\widehat{n}\left(\theta\right) \\ \frac{R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)}-1 & \text{if } \widehat{n}\left(\theta\right)\leq n<\overline{n} \\ 0-\frac{\ell}{(1-k)n} & \overline{n}\leq n\leq 1 \end{cases}, \tag{5}$$

where

$$\widehat{n}\left(\theta\right)=\frac{R\left(\theta\right)\left(1-\alpha\ell\right)-\left(1-k\right)r_2}{\left(1-k\right)\left[\frac{R(\theta)(1-\alpha\ell)}{\ell}-r_2\right]} \tag{6}$$

denotes the proportion of early withdrawals at which the bank exhausts the resources to repay the promised r_2 to the remaining debt holders at date 2, and

$$\overline{n}=\frac{\ell}{(1-k)} \tag{7}$$

is the proportion early withdrawals forcing the bank to liquidate the entire portfolio at date 1.

A debt holder's payoffs at date 1 and 2 are illustrated in Figure 1. At date 2, a debt holder obtains the promised repayment r_2 as long as bank resources suffice (i.e., for $n<\widehat{n}\left(\theta\right)$), while he obtains the pro-rata share $\frac{R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)}$ for $n\geq\widehat{n}\left(\theta\right)$. Similarly, at date 1, a debt holder receives the promised repayment $r_1=1$ as long as the bank can raise enough resources by prematurely liquidating its portfolio (i.e., for $n<\overline{n}$), while he receives the pro-rata share $\frac{\ell}{(1-k)n}$ for $n\geq\overline{n}$.

Insert Figure 1

In the range $[0,\overline{n})$, a debt holder's payoff differential $v\left(\theta,n\right)$ is weakly decreasing in n as long as

$$\ell<(1-k) \tag{8}$$

The condition (8), which is obtained by differentiating $v(\theta,n)$ with respect to n, means that the value of a bank's portfolio ℓ is not enough to repay $r_1 = 1$ if all debt holders were to withdraw at date 1 and thus guarantees that both $\widehat{n}(\theta)$ and $\bar{n}$ are lower than 1.

When condition (8) holds, debt holders' withdrawal decisions are strategic complements. As in Goldstein and Pauzner (2005), our model only exhibits the property of one-sided strategic complementarity since in the range $[\overline{n}, 1]$, a debt holder's incentive to roll over his debt claim until date 2 increases with n. This occurs because, when n is very large (i.e., $n \geq \overline{n}$), the more debt holders withdraw at date 1, the lower a debt holder's payoff from withdrawing at date 1, while the payoff at date 2 is zero. As in their framework, there exists a unique threshold equilibrium in which a debt holder withdraws if and only if his signal is below the threshold $s^*(k, \ell, r_2)$. At this signal value, the debt holder is indifferent between withdrawing at date 1 and rolling over his debt claim until date 2. The following result holds.

Proposition 1 *The model has a unique threshold equilibrium in which debt holders withdraw their debt claims at date* 1 *if they observe a signal below the threshold* $s^*(k, \ell, r_2)$ *and roll them over above. At the limit, when* $\varepsilon \to 0$, $s^*(k, \ell, r_2) \to \theta^*(k, \ell, r_2)$ *and corresponds to the solution to*

$$\int_0^{\widehat{n}(\theta)} r_2 dn + \int_{\widehat{n}(\theta)}^{\bar{n}} \frac{R(\theta)(1-\alpha\ell)\left[1 - \frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)} dn = \int_0^{\bar{n}} 1 dn + \int_{\bar{n}}^1 \frac{\ell}{(1-k)n} dn. \tag{9}$$

Thus, for any $\theta > \theta^*(k, \ell, r_2)$ *the bank is solvent and always repays the promised amounts.*

The proposition states that in the range of fundamentals $\theta \leq \theta^*(k, \ell, r_2)$, a bank fails as debt holders choose not to roll over their debt claims. By contrast, for any $\theta > \theta^*(k, \ell, r_2)$, all debt holders choose to roll over their debt claims and the bank is solvent in that it can repay the promised payment $\{r_1, r_2\}$ at date 1 and 2. At the threshold $\theta^*(k, \ell, r_2)$ debt holders are indifferent between rolling over and withdrawing at date 1 in that the expected payoff at date 2, as given by the two terms on the LHS in (9), equals that at date 1, as given by the terms in the RHS in (9).

The crisis threshold $\theta^*(k, \ell, r_2)$ emerges as the result of a coordination failure among debt holders. This is due to the existence of strategic complementarities in that each debt holder fears that other debt holders will withdraw and the bank will not have enough resources to repay their claims if he rolls over. As mentioned above, strategic complementarities emerge only if condition (8) holds. More formally, denoting as $k^{\max}(\ell)$ the solution to (8) with equality, $\theta^*(k, \ell, r_2)$ is the relevant crisis threshold if the bank has a level of capitalization k and portfolio liquidity ℓ in the region below the curve $k^{\max}(\ell)$. By contrast, when this is not the case, i.e., when (8) does not hold, there are no strategic complementarities among debt holders and the relevant crisis threshold is $\underline{\theta}(k, \ell)$.[3] The relevance of the two crisis thresholds is illustrated in Figure 2. In what follows, we refer to the crises in the region below $k^{\max}(\ell)$ as liquidity-driven ones and to those in the region above $k^{\max}(\ell)$ as solvency-driven crises.

Insert Figure 2

The following corollary illustrates how the levels of bank capitalization and portfolio liquidity affect the crisis thresholds for a given repayment r_2.

Corollary 1 *The following holds:*

1) The threshold $\underline{\theta}$ decreases with the level of bank capital k and increases with portfolio liquidity ℓ (i.e., $\frac{\partial \underline{\theta}}{\partial k} < 0$ and $\frac{\partial \underline{\theta}}{\partial \ell} > 0$);

2) The threshold θ^* i) decreases with k for any $k \in (\widetilde{k}(\ell), 1]$, and increases otherwise (i.e., $\frac{\partial \theta^*}{\partial k} < 0$ if $k \geq \widetilde{k}(\ell)$ and $\frac{\partial \theta^*}{\partial k} > 0$ otherwise); ii) decreases with ℓ for any $k \in (\underline{k}(\ell), \overline{k}(\ell))$ and $k \geq k^T(\ell)$, and increases otherwise (i.e., $\frac{\partial \theta^*}{\partial \ell} < 0$ if $\underline{k}(\ell) < k < \overline{k}(\ell)$ and $k \geq k^T(\ell)$ and $\frac{\partial \theta^*}{\partial \ell} > 0$ otherwise).

The boundaries $\widetilde{k}(\ell)$, $k^T(\ell)$, $\underline{k}(\ell)$ and $\overline{k}(\ell)$, with $\widetilde{k}(\ell) < \underline{k}(\ell) < \overline{k}(\ell)$, are defined in the Appendix.

The corollary, which is illustrated in Figure 3a and 3b, shows that capital is beneficial for fundamental crises, while portfolio liquidity is detrimental. The reason is that higher capital reduces leverage, thus leaving more resources to repay debt holders at date 2. By contrast,

[3] To keep the notation simple, in what follows, we denote the thresholds $\underline{\theta}(k, \ell)$ and $\theta^*(k, \ell, r_2)$ as $\underline{\theta}$ and θ^*, respectively.

liquidity reduces bank profitability at date 2, thus increasing the incentives of debt holders to withdraw at date 1.

Insert Figures 3a and 3b

The effect of capital and liquidity on the threshold θ^* is more involved as these affect debt holders' payoffs at both dates 1 and 2, as it emerges from (9). Higher capital increases debt holders' expected repayment both at date 1 and at date 2. The first effect dominates in the region below the curve $\widetilde{k}(\ell)$, where the bank faces a high risk of failure at date 1 as it is poorly capitalized and/or holds illiquid portfolios. The second effect dominates in the region above the curve $\widetilde{k}(\ell)$ since the bank is likely to withstand early withdrawals and survive until date 2.

Concerning portfolio liquidity, its increase raises a debt holder's repayment at date 1, but has an ambiguous effect on the date 2 payoff. On the one hand, higher liquidity reduces the (per unit) portfolio return at date 2; on the other hand, it lowers the amount of portfolio that has to be liquidated at date 1 so that more resources are available at date 2 to pay debt holders. As shown in the corollary, portfolio liquidity is only beneficial in the region between the curves $\underline{k}(\ell)$, $\overline{k}(\ell)$ and above $k^T(\ell)$. Here, the bank holds an intermediate level of capitalization and/or portfolio liquidity. As a consequence, it is less exposed to the risk of failure at date 1 than in the region below $\underline{k}(\ell)$ and $k^T(\ell)$, in which it holds either low levels of capital and liquidity or both, but there are still significant strategic complementarities among debt holders' actions. Thus, holding a more liquid portfolio allows the bank to liquidate fewer units at date 1 to repay withdrawing debt holders, and, as a consequence, increases the expected payoff for remaining investors at date 2.

By contrast, portfolio liquidity has a detrimental effect on the threshold θ^* in the regions below the curves $\underline{k}(\ell)$ and $k^T(\ell)$ and between $\overline{k}(\ell)$ and $k^{\max}(\ell)$. In the former region, the bank faces a high risk of failure at date 1, thus debt holders value the effect of higher liquidity on their date 1 payoff more than at date 2. In the latter region, liquidity increases θ^* because of the negative effect it has on the (per unit) portfolio return at date 2. Since in this region the bank faces a low risk of failure at date 1, as it holds high levels of capital and/or portfolio liquidity, the prospect of lower date 2 profitability increases debt holders' incentives to withdraw, thus increasing θ^*.

Corollary 1 has implications on the effects of capital and liquidity on the probability of crises. First, increasing capital or liquidity does not help banks that face high risk of failure at date 1 (as in the regions below $\widetilde{k}(\ell)$ and $\underline{k}(\ell)$ in Figures 3a and 3b) because they are poorly capitalized and hold illiquid portfolios. Second, and related to this, the timing of a regulatory/supervisory intervention is key: Asking banks to recapitalize and/or hold more liquidity when they face a high risk of failure at date 1 may precipitate a run rather than containing it because debt holders expect even higher payoffs at date 1. Third, holding more liquid portfolios may increase run risk for banks that are well capitalized and/or hold already enough liquid portfolios (as in the region above $\overline{k}(\ell)$ in Figure 3b) as this has mainly a negative effect on their date 2 profitability. In line with this, imposing the same regulation to banks with different balance sheets may not be optimal.

To sum up, the analysis of the properties of the crisis thresholds shows that the same increase in capital or liquidity may have different effects on stability for weakly, moderately or strongly capitalized banks, as well as for banks with low, moderate or high portfolio liquidity. The results highlight the importance of considering the interaction between capital and liquidity to assess their effects on stability. As we show below, looking at such interaction is crucial for the bank's choice of capital and portfolio liquidity, as well as for designing and evaluating capital and liquidity regulation.

3.2 Bank choice

Having computed the crisis thresholds $\underline{\theta}$ and θ^*, we can now characterize the bank's decisions concerning capital k, portfolio liquidity ℓ and debt interest rate r_2 at date 0. To do this, we start by assuming that (9) holds so that θ^* is the relevant crisis threshold and we then show that this is consistent with the bank's choice. Given this, the bank problem is as follows:

$$\max_{k,\ell,r_2} \Pi^B = \int_{\theta^*}^{1} \left[R(\theta)(1-\alpha\ell) - (1-k)\, r_2\right] d\theta - k\rho \tag{10}$$

subject to

$$IR^D : \int_0^{\theta^*} \frac{\ell}{1-k} d\theta + \int_{\theta^*}^{1} r_2 d\theta \geq 1, \tag{11}$$

$$\Pi^B \geq 0, \tag{12}$$

$$0 \leq k \leq 1,\ 0 \leq \ell \leq 1. \tag{13}$$

The bank chooses k, ℓ and r_2 to maximize its expected profit Π^B subject to a number of constraints. The first term in (10) is the revenue of the bank net of the debt repayments at date 2 when, for $\theta \geq \theta^*$, all debt holders roll over their debt and the bank remains solvent, while the second term represents the expected return to equity holders. Condition (11), which represents debt holders' participation constraint, requires their expected payoffs from lending to the bank to be at least equal to the storage return. For $\theta < \theta^*$, debt holders choose not to roll over their debt at date 1 thus forcing the liquidation of the entire bank portfolio for the value ℓ. Each debt holder then receives the pro-rata share of bank's available resources $\frac{\ell}{1-k}$. For $\theta \geq \theta^*$, debt is rolled over and investors receive the promised repayment r_2. The second constraint in (12) is a non-negativity constraint on bank profit, while the last two conditions in (13) are physical constraints on the level of capital and portfolio liquidity.

The solution to the bank's maximization problem yields the following result.

Proposition 2 *The market equilibrium features $r_2^B > 1$ as the solution to (11) and $k^B \in (0,1)$ and $\ell^B \in (0,1)$ as given by the solutions to*

$$-\left[\frac{\partial\theta^*}{\partial k} + \frac{\partial\theta^*}{\partial r_2}\frac{dr_2}{dk}\right]\left[R(\theta^*)(1-\alpha\ell) - \ell\right] - (\rho - 1) = 0, \tag{14}$$

and

$$-\left[\frac{\partial\theta^*}{\partial\ell}+\frac{\partial\theta^*}{\partial r_2}\frac{dr_2}{d\ell}\right]\left[R\left(\theta^*\right)(1-\alpha\ell)-\ell\right]+\int_0^{\theta^*}d\theta-\int_{\theta^*}^1\alpha R\left(\theta\right)d\theta=0. \tag{15}$$

The equilibrium pair $\left\{k^B,\ell^B\right\}$ *satisfies condition (8) and lies in the region between the curves* $\underline{k}\left(\ell\right)$ *and* $\overline{k}\left(\ell\right)$ *and* $k^T\left(\ell\right)$*.*

The proposition shows that in equilibrium the bank chooses to be exposed to liquidity crises due to strategic complementarities in debt holders' withdrawal decisions and to liquidate profitable portfolios prematurely, forgoing the return $R\left(\theta\right)(1-\alpha\ell)-\ell$. Increasing capital up to the level $\ell=(1-k)$ that is necessary to eliminate liquidity crises requires the bank to bear the higher cost of capital $\rho>1$, while entailing no benefit from preventing the premature liquidation of its portfolio. In other words, when k^B and ℓ^B are such that $\ell=(1-k)$, the benefit for the bank in terms of reduced crisis probability approaches zero, while the marginal cost in terms of higher funding costs (i.e., $\rho>1$) is still positive.[4] It follows that the bank chooses a level of capital that is too low relative to the socially optimal level, as we analyze in detail below.

At the optimum, the bank always chooses k^B and ℓ^B in the range where both $\frac{\partial\theta^*}{\partial k}<0$ and $\frac{\partial\theta^*}{\partial\ell}<0$ as this allows it to be less exposed to crises and, in turn, also reduce financing costs r_2^B, with an overall positive effect on its profit. In choosing its capital structure, the bank trades off the marginal benefit of capital with its marginal cost. The former, as represented by the first term in (14), is the gain in expected profits $\left[R\left(\theta^*\right)(1-\alpha\ell)-\ell\right]$ induced by a lower probability of a crisis, as measured by $\frac{\partial\theta^*}{\partial k}+\frac{\partial\theta^*}{\partial r_2}\frac{dr_2}{dk}$. The latter, as captured by the last two terms in (14), is the increase in funding cost $\rho-1$ associated with an increased reliance on equity financing.

The interpretation for (15) is analogous, with the only difference that the marginal benefit of an increase in liquidity consists now of two terms. First, more liquidity is beneficial as it leads to a lower crisis probability, as captured by the first term in (15). Second, more liquidity leads to a higher portfolio return at date 1, as captured by the second term in (15). Finally,

[4] When $(1-k)=\ell$ holds, $\theta^*\to\underline{\theta}$ and $r_2=1$ from debt holders' participation constraint in (11). Thus, $R\left(\theta^*\right)(1-\alpha\ell)-\ell=0$.

the marginal cost of increasing liquidity corresponds to the third term in (15), which captures the effect that more liquidity has on the date 2 return of bank portfolio.

3.3 Regulatory intervention

The market equilibrium characterized above is inefficient. Given the presence of costly capital (i.e., $\rho > 1$), the bank has an incentive to choose a level of capital k and a level of portfolio liquidity ℓ that are not optimal from a social welfare perspective for two reasons. First, in the unregulated equilibrium condition (8) holds. This implies that the relevant crisis threshold is $\theta^* > \underline{\theta}$ as defined in (4) so that when a run occurs for $\theta \leq \theta^*$, the bank liquidates its portfolio obtaining ℓ and thus giving up the return $R(\theta)(1 - \alpha\ell)$ at date 2. Such liquidation is inefficient for any $\theta > \theta^E$, where θ^E corresponds to the solution to

$$\frac{R(\theta)(1 - \alpha\ell)}{\ell} = 1. \tag{16}$$

Comparing (16) with (4), it follows from (8) that $\theta^E < \underline{\theta} < \theta^*$ and that $\theta^* \to \underline{\theta} = \theta^E$ when

$$\ell = 1 - k \tag{17}$$

holds. This implies that crises in the unregulated equilibrium can be inefficient, entailing a total loss TL equal to

$$TL = \int_{\theta^E}^{\theta^*} \left[R(\theta)(1 - \alpha\ell) - \ell\right] d\theta, \tag{18}$$

which is increasing in θ^*. Such a loss can be eliminated by enforcing $\theta^* \to \underline{\theta} = \theta^E$, which can be achieved, as shown in (17), by appropriately designing capital and liquidity requirements.

Second, as shown in Proposition 2, the bank chooses k and ℓ so to maximize its own expected profit rather than social welfare. The latter simply corresponds to the total return of bank portfolio, while the former also takes account of the cost of funding, and in particular, of the higher cost of capital ρ relative to the unitary cost of debt. This leaves room for regulation as a way to improve upon the market allocation, as we analyze next.

We consider a regulator that can set capital and/or liquidity requirements so to maximize social welfare as given by the sum of the bank's profits and investors' returns. Regulation is announced at the beginning of date 0, before the bank raises its funds and invests. Then, debt holders take their rollover decisions at date 1. We denote with the superscript R the equilibrium variables in the regulatory allocation.

The problem of the regulator is then to choose capital and/or liquidity requirements $\{k, \ell\}$ so to maximize

$$\begin{aligned} SW &= \int_{\theta^*}^{1} \left[R(\theta)(1-\alpha\ell) - (1-k)r_2\right] d\theta - k\rho + (1-k)\int_0^{\theta^*} \frac{\ell}{1-k} d\theta + (1-k)\int_{\theta^*}^{1} r_2 d\theta + k\rho = \\ &= \int_0^{\theta^*} \ell d\theta + \int_{\theta^*}^{1} R(\theta)(1-\alpha\ell)\, d\theta. \end{aligned} \tag{19}$$

subject to the promised repayment to debt holders r_2 satisfying (11) with equality as chosen by the bank and non-negative bank profit as in (12).

As shown in (19), the social welfare boils down to the expected portfolio returns as given by ℓ for $\theta \leq \theta^*$ when the bank liquidates its portfolio at date 1 to satisfy debt holders' withdrawals, and by $R(\theta)(1-\alpha\ell)$ for $\theta > \theta^*$ when no run occurs and the bank continues until date 2. Importantly, from a social perspective, both debt and equity have a cost of 1 since the repayments to debt holders and equity holders are a simple redistribution between the bank and investors. This means that there is a wedge between the private (i.e., the bank's) and the social cost of capital. We have the following result.

Proposition 3 *The regulator sets capital requirements k and the bank chooses the level of portfolio liquidity ℓ and the debt repayment r_2. Thus, the regulatory equilibrium features the following:*

i) For $\rho < \overline{\rho}$, $k^R = 1$, $\ell^R = 0$ and $\Pi^B(k^R, \ell^R) > 0$;

ii) For $\overline{\rho} \leq \rho < \widetilde{\rho}(\alpha)$, $k^R = 1 - \ell^R < 1$, and $\ell^R > 0$ as the solution to

$$\Pi^B(k^R, \ell^R) = 0,$$

and $r_2^R = 1$*;*

ii) Otherwise (i.e., for $\rho \geq \widetilde{\rho}(\alpha)$*),* $k^R < 1 - \ell^R < 1$ *as given by the solution to* $\Pi^B(k^R, \ell^R) = 0$*,* $\ell^R > 0$ *as the solution to*

$$-\left[\frac{\partial\theta^*}{\partial\ell} + \frac{\partial\theta^*}{\partial r_2}\frac{\partial r_2}{\partial\ell}\right]\left[R\left(\theta^*\right)(1-\alpha\ell) - \ell\right] + \int_0^{\theta^*} d\theta - \int_{\theta^*}^1 R(\theta)\alpha d\theta = 0, \tag{20}$$

and $r_2^R > 1$ *as the solution to (11).*

The thresholds $\overline{\rho}$*,* $\widetilde{\rho}(\alpha)$ *are defined in the Appendix.*

The main insight of Proposition 3 is that the regulator only needs to set capital requirements to tackle the inefficient liquidation of the bank portfolio, thus leaving the choice of portfolio liquidity to the bank. This is the case despite capital and liquidity requirements being equally effective in preventing the occurrence of liquidity crises. This result hinges on the fact that there is no wedge between the private and social cost of liquidity, while the opposite is true for capital. This implies that, for a given level of k, the bank chooses the socially optimal level of portfolio liquidity. In other words, the inefficiencies in the unregulated equilibrium arise because the bank chooses too much debt financing, which, in turn, forces it to hold an inefficiently high amount of liquidity to mitigate the run risk. Thus, requiring the bank to hold the level of capital that is optimal from a social perspective is enough to also enforce the socially optimal level of liquidity and so maximize social welfare.

The proposition, which is illustrated in Figure 4, also shows that the regulatory allocation depends on the costs of capital and liquidity, as measured by ρ and α. When the cost of capital is low enough (i.e., $\rho < \overline{\rho}$), the equilibrium features a corner solution in that the regulator requires banks to be fully equity financed. This implies that there is no short term debt and thus liquidity is unnecessary. The reason is that from a social perspective the additional cost of capital relative to debt is zero, whereby the return ρ is simply a redistribution between the bank and equity holders, while liquidity reduces the portfolio return at date 2.

Insert Figure 4

As capital becomes more costly, requiring banks to be fully equity financed is no longer feasible as it violates the bank's non-negative profit constraint. In this case, the regulatory allocation also features a positive level of portfolio liquidity. As long as the cost of capital and liquidity are contained, i.e., below the curve $\widetilde{\rho}(\alpha)$, there exist combinations of k and ℓ that are consistent with the bank making non-negative profit and, at the same time, eliminating inefficient crises. In other words, a pair $\left\{k^R, \ell^R\right\}$ satisfying $k^R = 1 - \ell^R$ and $\Pi^B\left(k^R, \ell^R\right) = 0$ emerges as the equilibrium of the regulated economy.

In the presence of higher cost of capital and liquidity (i.e., in the region above the curve $\widetilde{\rho}(\alpha)$), inefficient liquidity crises occur in equilibrium as eliminating them is either not feasible (i.e., it is not consistent with the bank making non-negative profits), or too costly for the bank as it would require it to hold excessively liquid portfolios. In this case, the regulator sets capital at the maximum level consistent with the bank's zero-profit constraint, while liquidity is chosen by bank at the level that maximizes bank's portfolio return in the presence of inefficient liquidity crises. The allocation with inefficient liquidity crises dominates the one with only efficient solvency crises when the cost of capital and liquidity are high, as it allows the regulator to choose a higher level of capital, consistent with the bank making non-negative profit. In other words, when the cost of capital and liquidity are high, the benefits in terms of prevention of inefficient liquidation are offset by the costs in terms of distortion to date 2 project return.

4 The economy with G banks

So far, we have focused on an economy with one bank in which the inefficiency spurs from the bank's preference for debt financing over equity and thus capital regulation is needed to restore optimality. Next, we extend our baseline framework to embed an additional friction that justifies the use of liquidity regulation and study its interaction with capital requirements. Specifically, we introduce the possibility of fire sales. To achieve this, we consider the presence of multiple symmetric banks, indexed with $g = 1, 2...G$, which share the same fundamental θ. As in the baseline model, each bank raises 1 unit of funds- a fraction k_g as capital and $1 - k_g$ as short term debt- to be invested in a risky portfolio, and chooses the liquidity ℓ_g of the portfolio. However, unlike in the baseline model, a bank may no longer obtain ℓ_g when liquidating its portfolio at date 1. The reason is that we now assume that banks sell their portfolios on a secondary market to outside investors holding in aggregate an amount $w > 0$, which we refer to as market liquidity. Thus, market conditions determine the price at which banks are now able to sell their portfolios. When market conditions are tight, there may be fire sales in that banks raise less than ℓ_g for each unit of their portfolio sold at date 1.

As common in the literature (see e.g., Shleifer and Vishny, 1992; Acharya and Yorulmazer, 2008; Acharya, Shin and Yorulmazer, 2010; and Eisenbach, 2017), we assume that the outside investors may be less able than banks in managing the portfolios they acquire so that transferring assets outside the banking sector may entail a loss of resources. In other words, each bank may be able to sell its portfolio for a value less than ℓ_g. The idea is that investors bear some costs in managing banks' assets and these costs increase with the aggregate degree of specificity of the assets as well as with the amount on sale. The specificity of bank assets is captured by the degree of liquidity of each bank portfolio ℓ_g in the sense that assets that are more liquid are less costly to be managed outside the banking sector, i.e., they are less specific. By contrast, the amount of assets on sale increases with the aggregate number of withdrawing depositors as represented by $\sum_g n_g(1 - k_g)$.

To ease the exposition, we denote as $\mathbf{n}$, $\mathbf{k}$ and $\boldsymbol{\ell}$ the vectors of proportions of running debt holders, bank capital and portfolio liquidity, respectively, and specify $Q(\mathbf{n}, \mathbf{k}, \boldsymbol{\ell})$ as a measure of the quantity and specificity of the pool of assets on sale in the market, with $\frac{\partial Q(.)}{\partial n_g} > 0$, $\frac{\partial Q(.)}{\partial k_g} < 0$ and $\frac{\partial Q(.)}{\partial \ell_g} < 0$ for all $g = 1, ...G$.[5] To capture the idea of fire sales due to asset redeployment,

[5] In what follows, the variables in bold always identify vectors.

we introduce the variable χ representing the market price of each unit of bank portfolio sold in the market at date 1, which can be specified as follows:

$$\chi(Q) = \begin{cases} \ell_g & \text{if } Q < \widehat{Q} \\ h(Q) < \ell_g & \text{if } Q \geq \widehat{Q} \end{cases}, \tag{21}$$

where $\widehat{Q} = \widehat{Q}(\mathbf{n}, \mathbf{k}, \boldsymbol{\ell}, w)$ is such that the market conditions become tight, i.e., $Q(\mathbf{n}, \mathbf{k}, \boldsymbol{\ell}) = w$. It follows from (21) that the amount that each bank can raise in the market depends on its own choice of portfolio liquidity ℓ_g as well as on the aggregate market funding conditions as affected by $\mathbf{n}, \mathbf{k}, \boldsymbol{\ell}$. Market conditions are good when either a few assets are sold in the market or the pool of assets on sale is not too specific relative to investors' resources (i.e., when $Q < \widehat{Q}$). In this case, each individual bank can raise ℓ_g per unit of portfolio liquidated at date 1. On the contrary when there is a large pool of assets on sale and/or they are very specific (i.e., $Q \geq \widehat{Q}$), fire sales emerge and banks can only raise (per unit) $\chi(Q) = h(Q) < \ell_g$. In what follows, we denote as $\underline{\chi}(\boldsymbol{\ell})$ the market price of each unit of bank portfolio at date 1 when all banks sell the entire portfolio in the secondary market.

This specification modifies the analysis in two important dimensions. First, it affects debt holders' rollover decisions, as it introduces strategic complementarity between banks in addition to that within a bank as characterized in the baseline model. Second, it leads to an additional inefficiency since, when G is large enough, banks do not internalize the effect that their individual choices have on the secondary market asset price.

We characterize the equilibrium in an economy with G banks following the same steps as in Section 3. We first characterize debt holders' rollover decisions at date 1 for given levels of capital k_g, portfolio liquidity ℓ_g and debt repayment r_{2g}. Then, we determine k_g, ℓ_g and r_{2g}. Finally, we derive the regulatory intervention that addresses the inefficiencies plaguing the unregulated economy.

4.1 Debt holders' rollover decisions

As in the baseline model, debt holders decide whether to withdraw at date 1 based on the signal s_i they receive since this provides information on both θ and other debt holders' actions. Unlike the baseline model, a debt holder in bank g is now concerned about the action taken by the other debt holders in his own bank as well as the action by debt holders in other banks.

In such context, there exist two types of strategic complementarity as in Goldstein (2005): *within*- and *between*-banks. The former refers to the fact that, as in the baseline model, a debt holder's incentive to withdraw at date 1 increases with the proportion of other debt holders in his own bank withdrawing at date 1. The latter captures the fact that a debt holder's incentive to withdraw at date 1 now also increases with the proportion of debt holders in other banks in the economy taking a similar action. The reason is that the more debt holders choose to withdraw early in other banks, the more those banks need to liquidate, thus leading to a larger drop in the market price $\chi(Q)$. This forces each bank to sell more assets, thus leaving fewer resources to repay the debt holders at date 2 and, as a result, increasing their incentive to run. Such *between*-banks strategic complementarity is a direct consequence of the existence of a common asset market, and its severity depends on the aggregate market conditions.[6]

As in the baseline model, each debt holder i in bank g receives a private signal of the same form as in (1) with the error term being now i.i.d. across both individuals and banks. Despite the existence of two types of strategic complementarities, as in the baseline model, debt holders choose to withdraw early when they receive a low enough signal and roll over otherwise. To see this formally, we follow the same steps as in Section 3.1 in that we first characterize the two extreme ranges of fundamentals where debt holders in each bank have a dominant action. The lower and upper dominance regions are the same as the ones in the baseline model, so that the threshold $\underline{\theta}$ and $\overline{\theta}$ remain the same. The reason is that $\underline{\theta}$ is computed under the assumption that no debt holders withdraw in any bank and $\overline{\theta}$ is obtained assuming that the bank's investment

[6] The strategic complementarities between banks resulting from the existence of a common asset market are also present in Eisenbach (2017) and Liu (2019). In these papers banks are exposed to idiosyncratic shocks to the fundamental θ_g rather than to aggregate shocks as in our framework.

becomes safe and yields $R(1)(1-\alpha\ell_g)$ also at date 1.

Besides these extreme ranges of θ, a debt holder's action depends crucially on what other debt holders, both within his own bank and in other banks, do. Thus, the existence of the *between*-banks strategic complementarity affects the derivations of the crisis threshold as illustrated in the following proposition.

Proposition 4 *For given* $\mathbf{k}$*,* $\boldsymbol{\ell}$*,* $\mathbf{r}_2$*, the model has a unique threshold equilibrium where all debt holders in bank* g *withdraw at date* 1 *if they receive a signal below* s_g^* *and roll over otherwise. The vector* $\mathbf{s}_G^*$ *of equilibrium threshold signals corresponds to the solution of the system of* G *indifference conditions of the form*

$$\int_0^{\widehat{n}\left(s_g^*,\mathbf{s}_{(-g)}^*\right)} r_{2g}dn_g + \int_{\widehat{n}(.)}^{\overline{n}\left(\mathbf{s}_{(-g)}^*\right)} \frac{R\left(s_g^*+\varepsilon-2\varepsilon n_g\right)(1-\alpha\ell_g)\left(1-\frac{n_g(1-k_g)}{\chi\left(Q\left(\boldsymbol{\ell},\boldsymbol{k},s_g^*,\mathbf{s}_{(-g)}^*\right)\right)}\right)}{(1-k_g)(1-n_g)}dn_g$$
$$-\int_0^{\overline{n}\left(\mathbf{s}_{(-g)}^*\right)} 1dn_g - \int_{\overline{n}\left(\mathbf{s}_{(-g)}^*\right)}^1 \frac{\chi\left(Q\left(\boldsymbol{\ell},\boldsymbol{k},s_g^*,s_{(-g)}^*\right)\right)}{(1-k_g)n_g}dn_g = 0 \tag{22}$$

for all $g=1,...G$*.*

The vector of equilibrium thresholds $\mathbf{s}_G^*$ is the solution of a system of G equations representing a debt holder's indifference condition between rolling over the debt until date 2 and withdrawing at date 1. It emerges from the expression (22) in the proposition that debt holders' actions crucially depend on the actions taken by all other debt holders in the economy, both in their own bank and in other banks. In particular, the greater the likelihood of a run in another bank (i.e., $s_{(-g)}^*$), the larger the probability of a run in bank g since the thresholds s_g^* and $s_{(-g)}^*$ are positively related. This implies that there are ranges of θ where debt holders run on their own bank only because they fear that runs are going to occur in other banks in the economy.

The positive correlation between the equilibrium thresholds of all G banks is a direct implication of the *between*-banks strategic complementarity induced by the existence of a common asset market. In an economy without fire sales, debt holders in different banks would take their withdrawal decisions only on the basis of other debt holders' actions in their own bank despite the common fundamental θ. Then, bank failures are (strategically) independent, that is the run at one bank does not affect the run risk at other banks. By contrast, in a framework with fire sales and *between*-banks strategic complementarity, there is contagion between banks as the run risk of one bank depends on that of the others. The following proposition describes this feature of the equilibrium with G banks.

Proposition 5 *When $\varepsilon \to 0$, all debt holders withdraw early if $\theta < \theta_G^*$ and roll over otherwise, with $\theta_G^* > \theta^*$ as characterized in the baseline model.*

The proposition shows that, similarly to the baseline model, there is a unique crisis threshold θ_G^* and this is the same for all banks. However, banks are now more fragile in that they fail in a larger range of fundamentals (i.e., $\theta_G^* > \theta^*$). In the region between θ^* and θ_G^* banks fail because of the strategic interdependence in the asset market, rather than because they share the same fundamental θ. In other words, each bank fails because its debt holders are concerned about the run risk at other banks in the system and the negative consequences it has for market funding conditions.

4.2 Banks' choices

Having characterized debt holders' withdrawal decisions and the likelihood of a bank failure at date 1, we can now turn to date 0 and solve banks' choices of capital k_g, liquidity ℓ_g and interest rate on debt r_{2g}. As banks are symmetric we focus on a representative bank, thus removing the subscript g, and we use the subscript G to denote the equilibrium variables.

The maximization problem is similar to that in Section 3.2 with only a few differences. First, the bank makes now positive profits only for $\theta > \theta_G^* > \theta^*$. Second, debt holders' participation constraint is now equal to

$$IR_G^D\text{:}\ \int_0^{\theta_G^*} \frac{\underline{\chi}(\ell)}{1-k} d\theta + \int_{\theta_G^*}^1 r_2 d\theta \geq 1, \tag{23}$$

where $\underline{\chi}(\ell)$ captures the amount the bank raises in the asset market when all banks are selling their entire portfolio at date 1. The function $\underline{\chi}(\ell)$ depends on the degree of specificity of the pool of assets on sale, as determined by all banks' liquidity choices ℓ, but it no longer depends on $\mathbf{k}$ and $\mathbf{n}$ as all banks liquidate the entire portfolio (i.e., 1 unit), with

$$\underline{\chi}(\ell) = \begin{cases} \ell & \ell \geq \widehat{\ell} \\ h(\ell) < \ell & \ell < \widehat{\ell} \end{cases}, \tag{24}$$

and $h'(\ell) > 0$. The expression in (24) hints to the fact that fire sales do not occur when banks hold sufficiently liquid portfolios (i.e.,when $\ell \geq \widehat{\ell}$), while they emerge otherwise (i.e., when $\ell < \widehat{\ell}$).[7] As in the baseline model, debt holders receive a pro-rata share of bank's available resources, $\frac{\underline{\chi}(\ell)}{1-k}$, in the event of a run and the promised repayment r_2 when no run occurs.

The solution to the bank's maximization problem yields the following result.

Proposition 6 *The market equilibrium features $r_{2G}^B > 1$ as the solution to (23) and $k_G^B \in (0,1)$ and $\ell_G^B \in (0,1)$ as given by the solutions to*

$$-\left[\frac{\partial\theta_G^*}{\partial k} + \frac{\partial\theta_G^*}{\partial r_2}\frac{dr_2}{dk}\right]\left[R(\theta_G^*)(1-\alpha\ell) - \underline{\chi}(\ell)\right] - \rho + 1 = 0, \tag{25}$$

and

$$-\left[\frac{\partial\theta_G^*}{\partial \ell} + \frac{\partial\theta_G^*}{\partial r_2}\frac{dr_2}{d\ell}\right]\left[R(\theta_G^*)(1-\alpha\ell) - \underline{\chi}(\ell)\right] - \int_{\theta^*}^{1} \alpha R(\theta)\, d\theta = 0. \tag{26}$$

In equilibrium liquidity crises occur since $\underline{\chi}(\ell) < \ell_G^B < \left(1 - k_G^B\right)$ holds.

As in the baseline model, banks choose to be exposed to liquidity crises. As before, the reason is that when $\underline{\chi}(\ell) = 1 - k$, the gain in terms of lower run probability approaches zero, while the loss in terms of higher financing costs is still equal to $\rho - 1$. Thus, reducing k slightly so that $\underline{\chi}(\ell) < 1 - k$ holds is always optimal.[8]

In choosing their capital structure k and portfolio liquidity ℓ, banks trade-off their marginal

[7] The exact value of $\widehat{\ell}$ depends on the tightness of the market, as measured, for example, by the amount of resources available to outside investors w: The larger w, the lower the $\widehat{\ell}$.

[8] When $1 - k = \underline{\chi}(\ell)$, the term $\left[R(\theta_G^*)(1-\alpha\ell) - \underline{\chi}(\ell)\right]$ simplifies to $\left[R(\underline{\theta})(1-\alpha\ell) - \underline{\chi}(\ell)\right] = 0$ since $\theta_G^* \to \underline{\theta}$, with $\underline{\theta}$ as given by (4).

benefits and costs. The former, which are captured by the first term in (25) and (26), refer to the reduction in the run threshold θ_G^* and in the bank financing costs associated with higher capital and liquidity. The latter, as given by $\rho - 1$ in (25) and the last term in (26), capture the higher financing costs and the lower return, respectively, triggered by a marginal increase in k and ℓ. Equation (25) is as in the case with one bank except that the value of the bank portfolio at date 1 is now $\underline{\chi}(\ell)$ instead of ℓ. By contrast, equation (26) also differs from the baseline model as banks take $\underline{\chi}(\ell)$ as given and thus do not internalize the effect of the choice of ℓ on the date 1 portfolio value $\chi(\ell)$.

4.3 Regulatory intervention

The market equilibrium characterized above entails three inefficiencies. First, as in the baseline model, banks' choice of k and ℓ spurs the occurrence of liquidity crises, thus leading to the premature termination of potentially profitable portfolios. Second, as in the baseline model, each bank chooses k and ℓ taking into account the wedge between the higher cost of capital ρ relative to the unitary cost of debt, which is zero from a social perspective. Third, banks do not internalize the effect that their liquidity choices have on the market funding conditions and thus on the market price $\underline{\chi}(\ell)$. In what follows, we consider the case where $\ell_G^B < \widehat{\ell}$ so that the unregulated equilibrium features fire sales, i.e., $\underline{\chi}(\ell) < \ell_G^B$. This is equivalent to assume that the market liquidity w is limited relative to the quality and quantity of banks' portfolios on sale. We can then specify the per bank output loss in the unregulated economy with G banks as follows:

$$TL = \int_0^{\theta_G^*} \left[\ell - \underline{\chi}(\ell)\right] d\theta + \int_{\theta^E}^{\theta_G^*} \left[R(\theta)(1 - \alpha\ell) - \ell\right] d\theta \tag{27}$$

The first term in (27) captures the loss associated with fire sales in the event of run for $\theta \leq \theta_G^*$, while the second term represents the output loss associated with the inefficient premature liquidation of portfolios for $\theta^E < \theta \leq \theta_G^*$, similarly to the case with one bank analyzed before.

As in the baseline model, the question is whether a regulator can improve upon the market allocation. Her maximization problem is as in the baseline model with the difference that in the event of a run the portfolio value at date 1 is $\underline{\chi}(\ell)$ as specified in (24) rather than ℓ. Thus, formally, the regulator chooses the level of bank capital k_G^R and portfolio liquidity ℓ_G^R at date 0 to maximize the per bank social welfare SW_G as given by:

$$\max_{k_G^R,\ell_G^R} SW_G = \int_0^{\theta_G^*} \underline{\chi}(\ell)\, d\theta + \int_{\theta_G^*}^1 R(\theta)(1-\alpha\ell)\, d\theta$$

subject to

$$\begin{aligned} r_2^B &= \arg\max \Pi^B, \\ \Pi^B &\geq 0,\, 0 \leq k_G^R \leq 1,\ 0 \leq \ell_G^R \leq 1, \end{aligned}$$

and

$$\underline{\chi}(\ell) \leq \ell.$$

The following proposition characterizes the optimal regulatory intervention.

Proposition 7 *The regulator sets both capital and liquidity requirements k, while banks choose the debt repayment r_2. Thus, the regulatory equilibrium features the following:*

i) For $\rho \leq \overline{\rho}$, $k_G^R = 1$, $\ell_G^R = 0$ and $\Pi^B(k_G^R, \ell_G^R) > 0$;

ii) For $\overline{\rho} < \underline{\rho}(\alpha) \leq \rho < \widetilde{\rho}(\alpha)$ and $\alpha \leq \overline{\alpha}$, $k_G^R = 1 - \ell_G^R$, $\ell_G^R \geq \widehat{\ell}$ and $r_{2G}^B = 1$;

iii) For $\rho < \underline{\rho}(\alpha)$ and $\alpha > \overline{\alpha}$, $k_G^R = 1 - \underline{\chi}(\ell)$ and $\ell_G^R < \widehat{\ell}$ as given by the solution to $\Pi^B(k_G^R, \ell_G^R) = 0$ and $r_{2G}^B = 1$

iii) Otherwise (i.e., for $\rho \geq \widetilde{\rho}(\alpha)$), $k^R < 1-\ell^R < 1$ as given by the solution to $\Pi^B(k_G^R, \ell_G^R) = 0$, $\ell^R > 0$ as the solution to

$$-\left[\frac{\partial\theta_G^*}{\partial\ell} + \frac{\partial\theta_G^*}{\partial r_2}\frac{\partial r_2}{\partial\ell}\right]\left[R(\theta^*)(1-\alpha\ell) - \underline{\chi}(\boldsymbol{\ell})\right] + \int_0^{\theta_G^*} \frac{\partial\underline{\chi}(\ell)}{\partial\ell} d\theta - \int_{\theta_G^*}^1 R(\theta)\alpha d\theta = 0, \tag{28}$$

and $r_{2G}^B > 1$ as the solution to (11).

The thresholds $\overline{\rho}$, $\widetilde{\rho}(\alpha)$, $\underline{\rho}(\alpha)$ and $\overline{\alpha}$ are defined in the Appendix.

The main insight of the proposition is that both capital and liquidity requirements are needed to tackle the inefficiencies of the market equilibrium. The reason is that there is now a wedge between private and social cost for both capital and liquidity. For capital, the wedge is as in the baseline model. For liquidity, the wedge emerges as its social cost is now lower than the private cost: Increasing liquidity reduces the extent of costly fire sales, as $\underline{\chi}(\ell)$ is increasing in ℓ, besides reducing the date 2 return of banks' investment projects. As a result, the characterization of the optimal regulation described in the proposition is more involved than in the baseline model.

The proposition, which is illustrated in Figure 5, shows that, as in the baseline model, the regulatory allocation depends on the costs of capital and liquidity, as measured by ρ and α. As in the economy with one bank, when the cost of capital is small (i.e., $\rho < \overline{\rho}$), the regulator finds it optimal to require banks to be fully equity financed. By doing so, it eliminates the run risk and so the need for liquidity. Thus, differences in the characterization of the regulatory equilibrium between the baseline model and the framework featuring fire sales only emerge when the cost of capital is large (i.e., $\rho \geq \overline{\rho}$). The proposition shows that when the cost of liquidity is high (i.e., $\alpha > \overline{\alpha}$) but that of capital is small (i.e., $\rho < \underline{\rho}(\alpha)$), the regulator chooses to eliminate liquidity-driven runs (i.e., $\theta^* \to \underline{\theta}$), but does not to require banks to hold a level of portfolio liquidity above $\widehat{\ell}$ so that fire sales still occur in equilibrium. The reason is that when the cost of liquidity is large eliminating fire sales is very costly in terms of forgone portfolio return at date 2. Still, though, the low cost of capital allows the regulator to require banks to raise a large fraction of their funds as equity, thus reducing the likelihood of a run and so the cost associated with the fire sales. Interestingly, in this case, the regulator eliminates liquidity crises, but, at the same time, prevents the liquidation of inefficient project. To see this, notice that when the regulator sets $k_G^R = 1 - \underline{\chi}(\ell) < 1 - \ell$, $\underline{\theta} < \theta^E$, which implies that for any $\theta \in [\underline{\theta}, \theta^E)$, runs do not occur and so portfolios yielding a date 2 return $R(\theta)(1 - \alpha\ell) < \ell$ are not liquidated.

Insert Figure 5

As the cost of capital increases and that of liquidity decreases, the regulator finds it optimal to eliminate both inefficient crises and fire sales (i.e., in the region between the curves $\underline{\rho}(\alpha)$ and $\widetilde{\rho}(\alpha)$ when $\alpha \leq \overline{\alpha}$). Finally, when the cost of capital and liquidity are very large (i.e., in the region above $\widetilde{\rho}(\alpha)$), similarly to the baseline model, the regulator finds it optimal not to eliminate liquidity crises, although fire sales are alleviated.

Overall, the proposition highlights that, when choosing the optimal regulatory mix, the regulator takes into account the impact that increased capital has on banks' profits, as well as the benefits and costs associated with increased portfolio liquidity. In particular, requiring banks to hold more liquidity has two effects. On the one hand, it is associated to a lower portfolio return at date 2. On the other hand, more liquidity increases the value of bank portfolio at date 1, by ameliorating fire sales. This implies that, when the former effect is large, the regulator may find it optimal not to eliminate fire sales, while the opposite is true when the cost of liquidity in terms of forgone date 2 return is more contained.

5 Concluding remarks

In this paper we develop a model where banks' exposure to crises depends on their balance sheet composition and both banks' and debt holders' decisions are endogenously determined. The paper offers a convenient framework to evaluate the implications of bank capital and liquidity on the likelihood of crises, as it allows to endogenize the probability of crises, distinguish their different type, and account for the different effects that changes in bank capital structure and portfolio liquidity have on each of them.

One of the main implications of the analysis is that, in order to be beneficial for stability, regulation should be designed considering both sides of banks' balance sheet. The same (marginal) increase in capital and liquidity may be beneficial for some banks, while detrimental for others. Real world regulatory tools like risk-weighted capital ratio (RWC), liquidity coverage ratio (LCR) or net stable funding ratio (NSFR) seem to fulfil this criterion, as they specify a ratio between banks' assets and liabilities (see Cecchetti and Kashyap, 2018).

The analysis of the impact of capital and liquidity on bank stability is also the starting point to characterize optimal regulation. In our framework, public intervention in the form of capital and liquidity requirements is desirable as the market equilibrium is plagued by two inefficiencies. First, banks choose levels of capitalization and portfolio liquidity that are consistent with the occurrence of liquidity crises and, as such, lead to inefficient portfolio liquidation. Second, in choosing their capital structure and portfolio liquidity banks do not fully internalize the effect that such choices have on social welfare, thus leading to lower aggregate output, as represented by lower long-term return of banks' investment projects, and costly fire sales.

We show that in the absence of fire sales, capital regulation is enough to tackle the inefficiencies of the market solution, while both capital and liquidity regulation are needed when the economy is also exposed to costly fire sales. In both cases, the ability and willingness of the regulator to eliminate the inefficiencies of the decentralized solution depend crucially on the cost of capital and liquidity. When they are contained, the regulator can achieve to eliminate all inefficiencies, while when they are large it may not be feasible or too costly in terms of forgone aggregate output to eliminate both inefficient crises and fire sales.

Our analysis of the impact of capital and liquidity on bank stability is conducted in a framework where the inefficiencies of the unregulated market equilibrium are all associated with the occurrence of runs and consequently the premature liquidation of banks' portfolio. In doing this, we disregard other possible sources of inefficiencies connected to the asset side decision of the bank (e.g., moral hazard associated with the riskiness of bank portfolios) which (capital) regulation is designed to tackle. Incorporating this into our analysis so to study the design of regulation in the presence of interaction between fire sales, run and credit risk is an interesting path for future research.

The analysis in our paper focuses on the interaction between two ex ante forms of intervention, namely capital and liquidity requirements. However, it abstracts from the interaction of those with other ex post policy tools– like, for example, the lender-of-last-resort policy and government guarantees to banks– which are used to limit financial instability (see, e.g., Rochet and Vives, 2004; Keister, 2016; and Allen, Carletti, Goldstein and Leonello, 2018). Analyzing the interaction between capital and liquidity regulation and other ex post interventions would require to assess and compare the effectiveness of each policy in preventing the occurrence of crises, as well as its costs in terms of forgone long-term returns and deadweight loss in the event of a crisis. We believe that Including this in our analysis to study the optimal policy mix represents a fruitful path for future research.

6 References

Acharya, V., and T. Yorulmazer (2008), "Cash-in-the-market pricing and optimal resolution of bank failures", *Review of Financial Studies*, 21, 2705-2742.

Acharya, V., Shin, H.S, and T. Yorulmazer (2010), "Crises resolution and bank liquidity", *Review of Financial Studies,* 24(6), 2166–2205.

Admati, A. and M. Hellwig (2013), The Bankers' New Clothes, Princeton: Princeton University Press.

Allen, F., Carletti, E., and R. Marquez (2011), "Credit market competition and capital regulation", *Review of Financial Studies*, 24(4), 993-1018.

Allen, F., Carletti, E., and R. Marquez (2015), "Deposits and capital structure", *Journal of Financial Economics*, 118(3), 601–619.

Allen, F., Carletti, E., Goldstein, I., and A. Leonello (2018), "Government guarantees and financial stability", *Journal of Economic Theory*, 177(C), 518-557.

Calomiris, C., and C., Kahn (1991), "The role of demandable debt in structuring optimal bank arrangements", *American Economic Review*, 81(3), 497-513.

Calomiris, C., Florian H., and M. Hoerova (2015), "A theory of bank liquidity requirements." Columbia Business School Research Paper 14-39 .

Carletti, E., R. Marquez, and S. Petriconi (2019), "The redistributive effects of bank capital regulation", *Journal of Financial Economics*, forthcoming.

Carlsson, H., and E. Van Damme (1993), "Global games and equilibrium selection." Econometrica: Journal of the Econometric Society, 989-1018.

Cecchetti, S., and A. Kashyap (2018), "What binds? Interactions between bank capital and liquidity regulations", in P. Hartmann, H. Huang and D. Schoenmaker (eds.) The Changing Fortunes of Central Banking, Cambridge, U.K.: Cambridge University Press, forthcoming.

Diamond, D.W. (1991), "Debt maturity structure and liquidity risk", *Quarterly Journal of Economics*, 106(3), 709-737.

Diamond, D.W., and P.H. Dybvig (1983), “Bank runs, deposit Insurance and liquidity”, *Journal of Political Economy*, 91(3), 401-419.

Diamond, D., and A. Kashyap (2016), “Liquidity requirements, liquidity choice and financial stability,” in Handbook of Macroeconomics

Vol 2B, John B. Taylor and Harald Uhlig (eds), North Holland.

Diamond, D.W., and R., Rajan (2001), "Liquidity risk, liquidity creation, and financial fragility: A theory of banking", *Journal of Political Economy*, 109(2), 287-327.

Eisenbach, T. (2017), "Rollover risk as market discipline: a two-sided inefficiency", *Journal of Financial Economics,* 126(2), 252-269.

Flannery, M. (1986), "Asymmetric information and risky debt maturity choice", *Journal of Finance*, 41(1), 19-37.

Goldstein, I. (2005), " Strategic Complementarities and the Twin Crises", *Economic Journal*, 115, 368-390.

Goldstein, I., and A. Pauzner (2005), "Demand deposit contract and probability of bank runs", *Journal of Finance*, 60(3), 1293-1328.

Goodhart, C.A.E., (1999), "Myths about the Lender of Last Resort", *International Finance*, 2, 339-360.

Harris, M., Opp, C., and M. Opp (2017), "Bank capital and the composition of credit", working paper, The Wharton School.

Hellmann, T. F., Murdock, K.C., and J. E. Stiglitz (2000), "Liberalization, moral hazard in banking, and prudential regulation: Are capital requirements enough?", *American Economic Review*, 90(1), 147-165.

Kashyap, A., Tsomocos, D., and A. Vardoulakis (2019), "Optimal bank regulation in the presence of credit and run risk", working paper 2017-097, Board of Governors of the Federal Reserve System.

Keister, T. (2016), "Bailouts and financial fragility", *Review of Economic Studies, 83,* 704-736.

König, P.J. (2015), "Liquidity requirements: A double-edged sword", *International Journal of Central Banking,* 11 (4), 129-168.

Macedo, D., and S., Vicente (2017), "Joint liquidity and capital regulation in a risk-shifting framework", mimeo.

Morris, S. and H.S. Shin (1998), "Unique equilibrium in a model of self-fulfilling currency attacks", *American Economic Review*, 88(3), 587-597.

Morris, S. and H.S., Shin (2003), "Global games: theory and applications" in M. Dewatripont, L. Hansen and S. Turnovsky (eds.), *Advances in Economics and Econometrics*, Cambridge University Press, Cambridge.

Repullo, R. (2004), "Capital requirements, market power, and risk-taking in banking", *Journal of Financial Intermediation,* 13(2), 156-182.

Rochet, J-C., and X. Vives (2004), "Coordination failures and the lender of last resort: was Bagehot right after all?", *Journal of the European Economic Association*, 2(6), 1116-1147.

Schepens, G. (2016), "Taxes and bank capital structure", *Journal of Financial Economics,* 120(3), 585-600.

Shleifer, A., and R.W. Vishny (1992), "Liquidation values and debt capacity: A market equilibrium approach." The Journal of Finance, 47.4,1343-1366.

Schilling, L. (2016), "Capital structure, liquidity and miscoordination on runs", mimeo.

Vives, X. (2014), "Strategic complementarity, fragility and regulation", *Review of Financial Studies*, 27(12), 3547-3592.

Walther, A. (2015), "Jointly optimal regulation of bank capital and liquidity", *Journal of Money, Credit and Banking*, 48(2), 415–448.

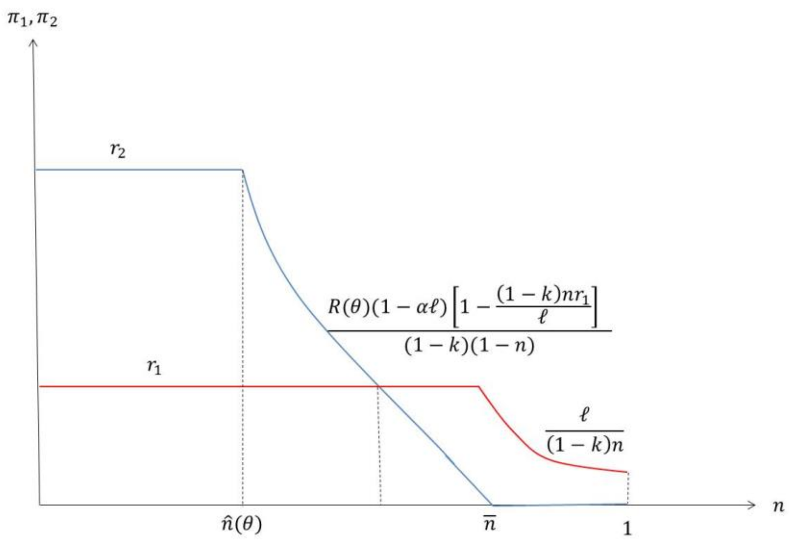

Figure 1: Debt holders' payoffs. The figure illustrates how a debt holder's payoffs change with the proportion of debt holders withdrawing their funds at date 1 n for a given $\theta \geq \underline{\theta}(k, \ell)$. The blue line represents a debt holder's payoff at date 2. A debt holder receives the promised repayment r_2 as long as the bank has enough funds to repay it (i.e., when $n < \hat{n}(\theta)$); otherwise he obtains a pro-rata share of bank's available resources. Such pro-rata is equal to zero when the bank liquidates the entire portfolio at date 1 (i.e., $n \geq \overline{n}$). The red line represents a debt holder's payoff at date 1. A debt holder receives the promised repayment r_1 as long as the bank has enough resources (i.e., $n < \overline{n}$); otherwise he obtains a pro-rata share.

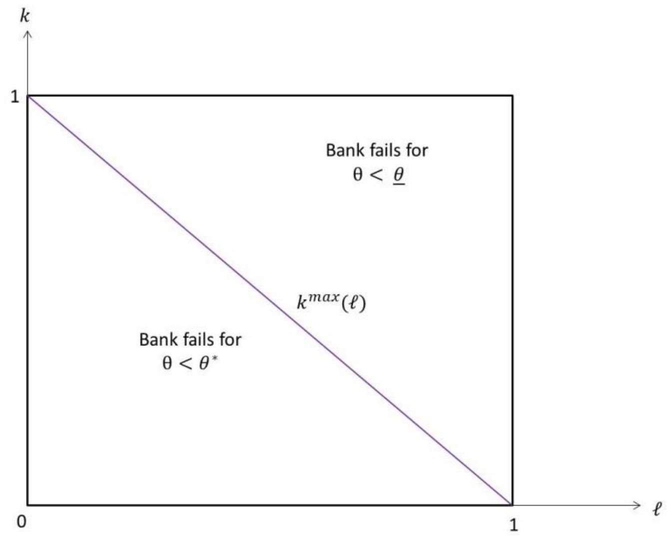

Figure 2: Capital, liquidity and type of crises. The figure illustrates how a bank's exposure to crises depends on its capital structure k and portfolio liquidity ℓ. A bank characterized by high capital and/or portfolio liquidity (i.e., one falling in the region above the curve $k^{max}(\ell)$) is only exposed to solvency-driven crises so that the relevant crisis threshold is $\underline{\theta}$. A bank characterized by low capital and/or portfolio liquidity (i.e., one falling in the region below the curve $k^{max}(\ell)$) is also exposed to liquidity-driven crises so that the relevant crisis threshold is θ^*. The curve $k^{max}(\ell)$ corresponds to $(1-k) = \ell$ and pins down the pairs $\{k, \ell\}$ for which there is no strategic complementarity among debt holders' withdrawal decisions.

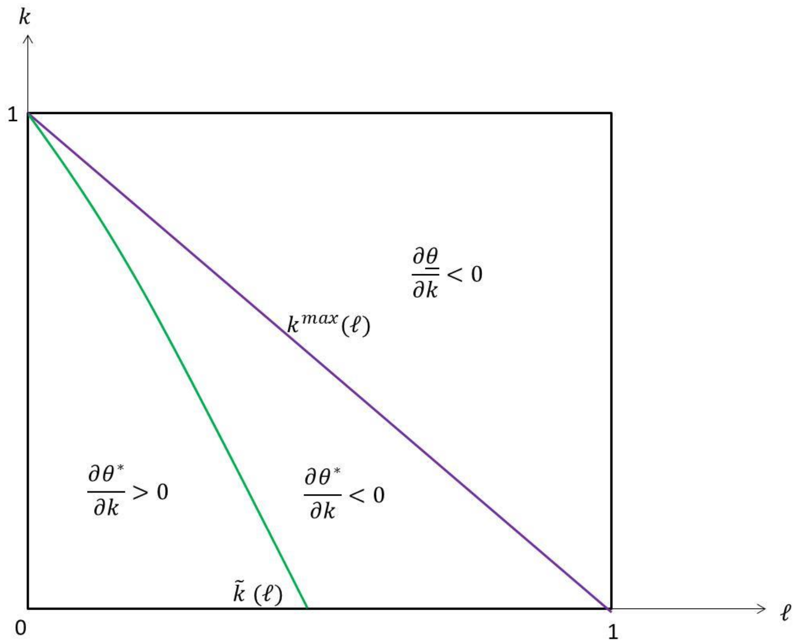

Figure 3a: Capital and Stability. The figure illustrates the effect of a marginal increase in bank capital on stability. Capital has a beneficial effect on stability for a bank characterized by intermediate and high values of capital and/or portfolio liquidity (i.e., $\frac{\partial \theta^*}{\partial k} < 0$ in the region above the curve $\tilde{k}(\ell)$), while it is detrimental otherwise.

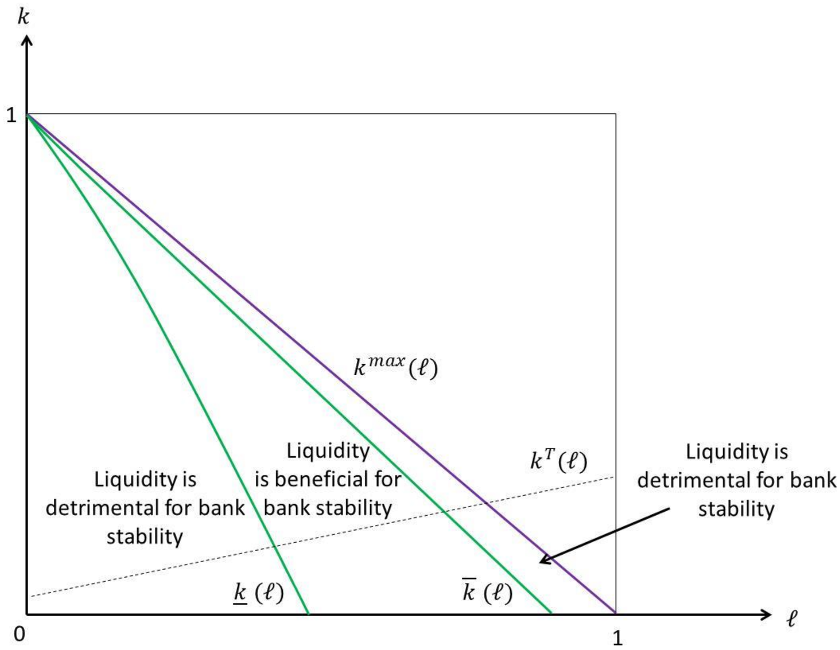

Figure 3b: Portfolio liquidity and stability. The figure illustrates the effect of a marginal increase in portfolio liquidity on stability. Liquidity has a beneficial role on stability only for banks characterized by intermediate values of capital and/or portfolio liquidity (i.e., $\frac{\partial \theta^*}{\partial \ell} < 0$ in the region bounded by the curves $\overline{k}(\ell), \underline{k}(\ell)$ and $k^T(\ell)$), while it is detrimental otherwise.

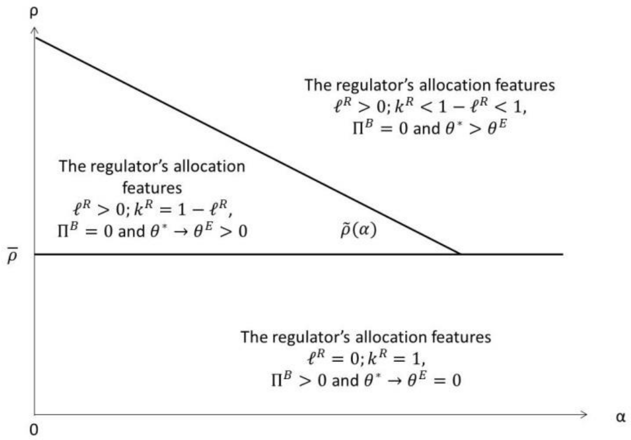

Figure 4: Regulatory equilibrium. The figure illustrates how the regulatory equilibrium varies depending on the cost of capital ρ and liquidity α. When the cost of capital is small, i.e., below $\overline{\rho}$, the regulator finds it optimal to require the bank to be fully equity-financed, i.e., $k^R = 1$ and $\ell^R = 0$. By doing so, she completely eliminates runs and so the need for costly liquidity. When the cost of capital is large (i.e., $\rho \geq \overline{\rho}$), requiring the bank to raise only capital is no longer feasible. As long as the cost of capital and liquidity are not too large (i.e., in the region below the curve $\tilde{\rho}(\alpha)$) eliminating inefficient runs by requiring the bank to hold $k^R = 1 - \ell^R$ with $\ell^R > 0$ is optimal. When the cost of capital and liquidity are higher (i.e., in the region above the curve $\tilde{\rho}(\alpha)$), this is no longer the case and the regulator finds it optimal to choose $k^R < 1 - \ell^R$ with $\ell^R > 0$.

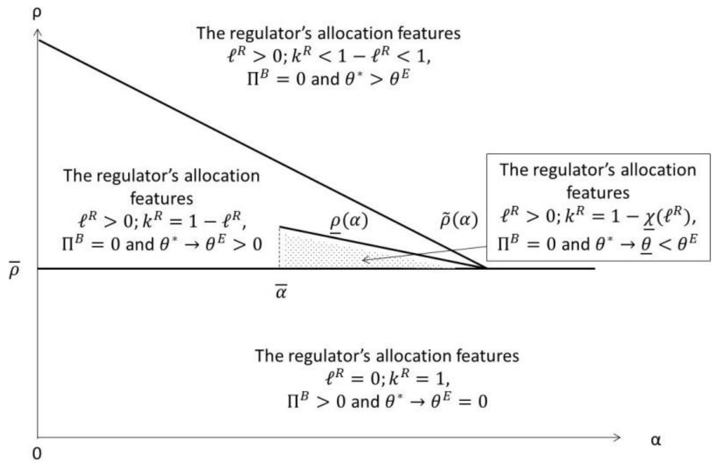

Figure 5: Regulatory equilibrium with G banks. The figure illustrates how the regulatory equilibrium varies depending on the cost of capital ρ and liquidity α. When the cost of capital is small, i.e., below $\overline{\rho}$, the regulator finds it optimal to require banks to be fully equity-financed, i.e., $k^R = 1$ and $\ell^R = 0$. By doing so, she completely eliminates runs and so the need for costly liquidity. When the cost of capital is large (i.e., $\rho \geq \overline{\rho}$), requiring banks to raise only capital is no longer feasible. As long as the cost of capital and liquidity are not too large (i.e., in the region below the curve $\tilde{\rho}(\alpha)$) eliminating panic runs is optimal. Otherwise (i.e., in the region above the curve $\tilde{\rho}(\alpha)$), the regulator finds it optimal to choose $\ell^R > 0$ and $k^R < 1 - \ell^R$ so that panic runs occur. In the region below the curve $\tilde{\rho}(\alpha)$, eliminating also fire sales by requiring the bank to hold $\ell^R \geq \hat{\ell}$ is optimal as long as the cost of liquidity α is small (i.e., and for $\alpha \leq \overline{\alpha}$). In the dotted area, below the curve $\underline{\rho}(\alpha)$ the regulator finds it optimal to set $k^R = 1 - \chi(\ell)$ and $\ell^R < \hat{\ell}$ so that in equilibrium fire sales still occur.

7 Online Appendix

Proof of Proposition 1: The proof follows closely that in Goldstein and Pauzner (2005) since our model also exhibits the property of *one-sided* strategic complementarity.

Assume that debt holders behave accordingly to a threshold strategy, that is each debt holder withdraws at date 1 if he receives a signal below s^* and rolls over otherwise. Then, the fraction of debt holders not rolling over the debt claim n is equal to the probability of receiving a signal below s^*. Given that debt holders' signals are independent and uniformly distributed in the range $[\theta - \varepsilon, \theta + \varepsilon]$, $n(s^*, \theta)$ is equal to

$$n(s^*, \theta) = \begin{cases} 1 & \text{if } \theta \leq s^* - \varepsilon \\ \frac{s^* - \theta + \varepsilon}{2\varepsilon} & \text{if } s^* - \varepsilon \leq \theta \leq s^* + \varepsilon \\ 0 & \text{if } \theta \geq s^* + \varepsilon \end{cases}. \tag{29}$$

When θ is lower than $s^* - \varepsilon$, all $(1-k)$ debt holders receive a signal below s^* and so withdraw at date 1, i.e., $n = 1$. On the contrary, when θ is higher than $s^* + \varepsilon$, all $(1-k)$ debt holders receive a signal above s^* and, as a result, decide to roll over their debt claim, i.e., $n = 0$. In the intermediate range of fundamental, when $s^* - \varepsilon \leq \theta \leq s^* + \varepsilon$, there is a partial run, in that only some debt holders withdraw at date 1. The proportion of those not rolling over their debt claim decreases linearly with θ, as fewer investors observe a signal below the threshold s^*.

Denote as $\Delta(s_i, \dot{n}(\theta))$ an agent's expected difference in utility between withdrawing at date 2 and at date 1 when he holds beliefs $\dot{n}(\theta)$ regarding the number of depositors running. The function $\Delta(s_i, \dot{n}(\theta))$ is given by

$$\Delta(s_i, \dot{n}(\theta)) = \frac{1}{2\varepsilon} \int_{s_i - \varepsilon}^{s_i + \varepsilon} E_n \left[v(\theta, \dot{n}(\theta)) \right] d\theta.$$

Since for any realization of θ, the proportion of depositors running is deterministic, we can write $n(\theta)$ instead of $\dot{n}(\theta)$ and the function $\Delta(s_i, \dot{n}(\theta))$ simplifies to

$$\Delta(s_i, n(\theta)) = \frac{1}{2\varepsilon} \int_{s_i - \varepsilon}^{s_i + \varepsilon} v(\theta, n(\theta)) d\theta.$$

Notice that when all depositors behave according to the same threshold strategy s^*, $n(\theta) = n(\theta, s^*)$ defined in (29). The following lemma states a few properties of the function $\Delta(s_i, \dot{n}(\theta))$.

Lemma 1 *i) The function $\Delta(s_i, \dot{n}(\theta))$ is continuos in s_i; ii) for any $a > 0$, $\Delta(s_i + a, \dot{n}(\theta) + a)$ is non-decreasing in a, iii) $\Delta(s_i + a, \dot{n}(\theta) + a)$is strictly increasing in a if there is a positive probability that $n < \overline{n}$ and $\theta < \overline{\theta}$.*

Proof of Lemma 1: The proof follows Goldstein and Pauzner (2005). The function $\Delta(.)$ is continuous in s_i, as s_i only changes the limits of integration in the formula for $\Delta(s_i, n(s^*, \theta))$. To show that the function $\Delta(s_i, n(s^*, \theta))$ is non-decreasing in a, we need first to show that $v(\theta, n)$ is non-decreasing in θ. As θ increases, we have two effects. First, a higher θ implies that $R(\theta)$ is higher, thus increasing the date 2 payoff in the range $\widehat{n}(\theta) \leq n < \overline{n}$. Second, a change in θ affects the threshold $\widehat{n}(\theta)$ as follows:

$$
\begin{aligned}
\frac{\partial \widehat{n}(\theta)}{\partial \theta} &= R'(\theta)(1-\alpha\ell)\frac{\left[\frac{R(\theta)(1-\alpha\ell)}{\ell} - r_2\right] - \left[R(\theta)(1-\alpha\ell) - (1-k)r_2\right]\frac{1}{\ell}}{(1-k)\left[\frac{R(\theta)(1-\alpha\ell)r_1}{\ell} - r_2\right]^2} = \\
&= \frac{R'(\theta)(1-\alpha\ell)}{(1-k)\left[\frac{R(\theta)(1-\alpha\ell)}{\ell} - r_2\right]^2} r_2 \left[\frac{(1-k)}{\ell} - 1\right] > 0,
\end{aligned}
$$

since $R'(\theta) > 0$ and $(1-k) > \ell$. Thus, as the interval $[0, \widehat{n}(\theta))$ where the utility differential $v(\theta, n) = r_2 - 1 > 0$ becomes larger, while the range $(\overline{n}, 1]$ is unaffected by a change in θ, the date 2 payoff increases so that the utility differential $v(\theta, n)$ is non-decreasing in θ. This also implies that $\Delta(s_i + a, \dot{n}(\theta) + a)$ is non-decreasing in a, as when a increases, debt holders see the same distribution of n but expects θ to be larger. In order for $\Delta(s_i + a, \dot{n}(\theta) + a)$ to be strictly increasing in a, we need that $\theta < \overline{\theta}$ and that there is a positive probability that $n < \overline{n}$. This is the case because, when $n < \overline{n}$ and $\theta < \overline{\theta}$, $v(\theta, n)$ is strictly increasing in θ, and, thus, $\Delta(s_i + a, \dot{n}(\theta) + a)$ is strictly increasing in a. ■

Since the rest of the proof follows closely that in Goldstein and Pauzner (2005) we omit it here and only specify the condition pinning down the threshold s^*. A debt holder who receives the signal s^* is indifferent between rolling over the debt claim until date 2 and withdrawing it at date 1. The threshold s^* can be computed as the solution to

$$
f(\theta, k, \ell) = \int_0^{\widehat{n}(s^*)} r_2 dn + \int_{\widehat{n}(s^*)}^{\overline{n}} \frac{R(\theta(n))(1-\alpha\ell)\left[1 - \frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)} dn - \int_0^{\overline{n}} 1 dn - \int_{\overline{n}}^1 \frac{\ell}{(1-k)n} dn = 0, \tag{30}
$$

where from (29), we obtain $\theta(n) = s^* + \varepsilon - 2\varepsilon n$ and $\widehat{n}(s^*)$ solves $R(\theta(n))(1-\alpha\ell)\left[1 - \frac{(1-k)n}{\ell}\right] - (1-k)(1-n)r_2 = 0$. At the limit, when $\varepsilon \to 0$, $\theta(n) \to s^*$ and we denote it as θ^*, which corresponds to the solution to the condition (9) in the proposition.

To complete the proof, we need to show that the bank is solvent for any $\theta > \theta^*$. To do that we need to exclude the possibility that the bank fails at date 2 despite all debt holders rolling over the debt until date 2. Denote as $\widehat{\theta}$ the level of fundamental at which the bank fails at date

2 even when all debt holders roll over the debt claim (i.e., when $n = 0$). The threshold $\widehat{\theta}$ solves

$$R(\theta)(1-\alpha\ell) - (1-k)r_2 = 0.$$

In order to show that the bank is always solvent for any $\theta > \theta^*$, we need to show that the threshold θ^* characterized in (30) larger than $\widehat{\theta}$. To see this, denote as $\widetilde{\theta}$ the level of θ at which the bank is at the margin between failing and being solvent at date 2 when n debt holders withdraw early. Then, $\widetilde{\theta}$ is the solution to

$$R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)\,n(s^*,\theta)}{\ell}\right] - (1-k)(1-n(s^*,\theta))\,r_2 = 0, \tag{31}$$

where $n\left(s^*, \widetilde{\theta}\right)$ is given in (29). Rearranging (31) as

$$R(\theta)(1-\alpha\ell) - (1-k)r_2 - n(s^*,\theta)\left[\frac{R(\theta)(1-\alpha\ell)(1-k)}{\ell} - (1-k)r_2\right] = 0,$$

it is easy to see that (31) is negative when evaluated at $\theta = \widehat{\theta}$ when $(1-k) > \ell$ holds. Thus, since (31) is increasing in θ, it follows that $\widetilde{\theta} > \widehat{\theta}$.

The equilibrium in debt holders' withdrawal decision characterized in the proposition corresponds to the pair $\{s^*, \theta^*\}$ solving (31) and the indifference condition as given by $v(\theta, n) = 0$ after the change of variable giving $\theta(n) = s^* + \varepsilon - 2\varepsilon n$. Thus, it is the case that, when $\varepsilon \to 0$, $s^* \to \theta^* > \widehat{\theta}$ and the proposition follows. □

Proof of Corollary 1: The proof proceeds in steps. First, we compute the effect of k and ℓ on the crisis threshold $\underline{\theta}$ and then their effect on the threshold θ^*.

Denote as $z(\theta, k, \ell) = 0$ the condition pinning down the threshold $\underline{\theta}(k, \ell)$ as given in (4). By using the implicit function theorem, we have that

$$\frac{\partial\underline{\theta}(k,\ell)}{\partial k} = -\frac{\frac{\partial z(\theta,k,\ell)}{\partial k}}{\frac{\partial z(\theta,k,\ell)}{\partial \theta}} \text{ and } \frac{\partial\underline{\theta}(k,\ell)}{\partial \ell} = -\frac{\frac{\partial z(\theta,k,\ell)}{\partial \ell}}{\frac{\partial z(\theta,k,\ell)}{\partial \theta}}.$$

The denominator $\frac{\partial z(\theta,k,\ell)}{\partial\theta} = \frac{R'(\underline{\theta})(1-\alpha\ell)}{1-k} > 0$ as $R'(\theta) > 0$. Thus, the sign of $\frac{\partial\underline{\theta}(k,\ell)}{\partial k}$ and $\frac{\partial\underline{\theta}(k,\ell)}{\partial \ell}$ are equal to the opposite sign of the respective numerators. Deriving (4) with respect to k and ℓ we obtain

$$\frac{\partial z(\theta,k,\ell)}{\partial k} = \frac{R(\underline{\theta})(1-\alpha\ell)}{(1-k)^2} > 0, \quad \frac{\partial z(\theta,k,\ell)}{\partial \ell} = -\frac{R(\underline{\theta})\,\alpha}{(1-k)} < 0,$$

which imply $\frac{\partial \underline{\theta}(k,\ell)}{\partial k} < 0$ and $\frac{\partial \underline{\theta}(k,\ell)}{\partial \ell} > 0$.

Consider now the effect of capital and liquidity on the threshold θ^*. Denote as $g(\theta, k, \ell) = 0$ the equation pinning down θ^*, as defined in (9). Using the implicit function theorem we obtain:

$$\frac{\partial \theta^*}{\partial k} = -\frac{\frac{\partial g(\theta,k,\ell)}{\partial k}}{\frac{\partial g(\theta,k,\ell)}{\partial \theta}} \text{ and } \frac{\partial \theta^*}{\partial \ell} = -\frac{\frac{\partial g(\theta,k,\ell)}{\partial \ell}}{\frac{\partial g(\theta,k,\ell)}{\partial \theta}},$$

The denominator $\frac{\partial g(\theta,k,\ell)}{\partial \theta}$ is given by

$$\frac{\partial g(\theta, k, \ell)}{\partial \theta} = \int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R'(\theta^*)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)} dn > 0$$

since the derivatives of the extremes of the integrals cancel out. Thus, the sign of $\frac{\partial \theta^*}{\partial k}$ and $\frac{\partial \theta^*}{\partial \ell}$ are equal to the opposite sign of $\frac{\partial g(\theta,k,\ell)}{\partial k}$ and $\frac{\partial g(\theta,k,\ell)}{\partial \ell}$, respectively.

We start from $\frac{\partial g(\theta,k,\ell)}{\partial k}$. Deriving (9) with respect to k and multiplying it by -1, we obtain

$$\frac{1}{(1-k)^2}\left[-\int_{\widehat{n}(\theta^*)}^{\overline{n}} \frac{R(\theta^*)(1-\alpha\ell)}{(1-n)} dn + \ell \int_{\overline{n}}^{1} \frac{1}{n} dn\right], \tag{32}$$

since the derivatives of the extremes of the integrals cancel out. Similarly, differentiating (9) with respect to ℓ, after a few manipulation and multiplying it by -1, we obtain

$$\frac{1}{(1-k)\ell}\left[\int_{\widehat{n}(\theta^*)}^{\overline{n}} \alpha\ell \frac{R(\theta^*)}{1-n} dn + \ell \int_{\overline{n}}^{1} \frac{1}{n} dn - \int_{\widehat{n}(\theta^*)}^{\overline{n}} \frac{R(\theta^*)(1-k)n}{(1-n)\ell} dn\right]. \tag{33}$$

as the derivatives of the extremes of the integrals cancel out.

Consider first the effect of k on θ^*. We can rearrange the terms in the square bracket in (32) as follows:

$$R(\theta^*)(1-\alpha\ell)\, Log\left[\frac{1-\overline{n}}{1-\widehat{n}(\theta^*)}\right] - \ell Log[\overline{n}]. \tag{34}$$

The first term is negative since $\overline{n} > \widehat{n}(\theta^*)$ and so $\frac{1-\overline{n}}{1-\widehat{n}(\theta^*)} < 1$, while the second one is positive since $\overline{n} < 1$. Using $\overline{n} = \frac{\ell}{(1-k)}$ and $\widehat{n}(\theta^*) = \frac{R(\theta^*)(1-\alpha\ell)-(1-k)r_2}{(1-k)\left[\frac{R(\theta^*)(1-\alpha\ell)}{\ell}-r_2\right]}$, after a few manipulations, the expression above can be rewritten as follows:

$$Log\left[\left(1-\frac{\ell r_2}{R(\theta^*)(1-\alpha\ell)}\right)^{\frac{R(\theta^*)(1-\alpha\ell)}{\ell}}\right] - Log\left[\left(\frac{\ell}{(1-k)}\right)\right]. \tag{35}$$

Denote as $\widetilde{k}(\ell)$ the solution to $Log\left[\left(1-\frac{\ell r_2}{R(\theta^*)(1-\alpha\ell)}\right)^{\frac{R(\theta^*)(1-\alpha\ell)}{\ell}}\right]-Log\left[\left(\frac{\ell}{(1-k)}\right)\right]=0$. The expression in (35) can be rearranged as

$$\widetilde{k}(\ell)=1-\ell(\Lambda)^{-\frac{R(\theta^*)(1-\alpha\ell)}{\ell}}, \tag{36}$$

where $\Lambda=\left(1-\frac{\ell r_2}{R(\theta^*)(1-\alpha\ell)}\right)$. Since for any pair $\{k,\ell\}$, θ^* varies between $\underline{\theta}$ and $\overline{\theta}\to 1$, it holds that $\widetilde{k}(\ell)<k^{\max}(\ell)$ for any $\ell\in(0,1)$, since $k^{\max}(\ell)=1-\ell$ and $(\Lambda)^{-\frac{R(\theta^*)(1-\alpha\ell)}{\ell}}>1$. Furthermore, from (36), it follows that $\widetilde{k}(\ell)\to 1$, when $\ell\to 0$ and that $\widetilde{k}(\ell)=0$ requires $\ell>0$.

Consider a pair $\{k,\ell\}$ in the region below $k^{\max}(\ell)$. When we approach the curve $k^{\max}(\ell)$, the threshold $\theta^*\to\underline{\theta}$. To see this, we can rearrange the expression in (9) as follows:

$$\int_0^{\widetilde{n}(\theta)}\left[\min\left\{r_2,\frac{R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)}\right\}-1\right]dn+ \tag{37}$$
$$+\int_{\widetilde{n}(\theta)}^{\bar{n}}\left[\frac{R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)}-1\right]dn-\int_{\bar{n}}^{1}\frac{\ell}{(1-k)n}dn,$$

with $\overline{n}=\frac{\ell}{(1-k)}$ and $\widetilde{n}(\theta)=\frac{R(\theta)(1-\alpha\ell)-(1-k)}{(1-k)\left[\frac{R(\theta)(1-\alpha\ell)}{\ell}-1\right]}$ denoting the proportion of debt holders withdrawing at date 1 at which the bank's resources at date 2 are exactly enough to pay 1 to debt holders rolling over the debt claim until date 2. When $k\to k^{\max}(\ell)$, $\overline{n}\to\widetilde{n}(\theta)\to 1$ and the expression above simplifies to

$$\int_0^1\left[\min\left\{r_2,\frac{R(\theta)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)}\right\}-1\right]dn=0.$$

Since $r_2>r_1$, θ^* solves $\frac{R(\theta)(1-\alpha\ell)}{(1-k)}-1=0$, which is equivalent to the equation pinning down $\underline{\theta}$, as given in (4).

This implies that for pairs $\{k,\ell\}$ very close to the curve $k^{\max}(\ell)$, $\frac{\partial\theta^*}{\partial k}<0$. Thus, since $\frac{\partial\theta^*}{\partial k}=0$ on the curve $\widetilde{k}(\ell)$, by continuity it must be the case that $\frac{\partial\theta^*}{\partial k}<0$ in the region between $\widetilde{k}(\ell)$ and $k^{\max}(\ell)$.

Consider now a pair $\{k,\ell\}$ below the curve $\widetilde{k}(\ell)$ and close to the axes origin. For any $0\leq k<<1$ and $\ell\to 0$, the expression in (35) is positive since the second term approaches to

$-\infty$, while the first term is equal to

$$Lim_{\ell \to 0} \frac{R(\theta^*)(1-\alpha\ell)}{\ell} Log[\Lambda] = Lim_{\ell \to 0} \frac{Log[\Lambda]}{\frac{\ell}{R(\theta^*)(1-\alpha\ell)}},$$

and using l' Hopital's rule, after a few manipulations, we obtain

$$Lim_{\ell \to 0} \frac{R(\theta^*)(1-\alpha\ell)}{\ell} Log[\Lambda] = -\frac{\frac{r_2}{Lim_{\ell \to 0}[R(\theta^*)(1-\alpha\ell)-\ell r_2](1-\alpha\ell)}}{\frac{1}{Lim_{\ell \to 0}\left[R(\theta^*)(1-\alpha\ell)^2\right]}} = -r_2 < 0,$$

where $Lim_{\ell \to 0}[R(\theta^*)(1-\alpha\ell)]$ is equal to a finite number. This implies that $\frac{\partial\theta^*}{\partial k} > 0$ for $k << 1$ and $\ell \to 0$. Since the derivative $\frac{\partial\theta^*}{\partial k}$ is zero on the curve $\widetilde{k}(\ell)$, by continuity it stays positive below $\widetilde{k}(\ell)$.

Consider now the effect of liquidity ℓ on θ^*. The expression (33) determining the sign of $\frac{\partial\theta^*}{\partial\ell}$ can be rearranged as follows, after adding and subtracting $\frac{1}{(1-k)\ell}\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)}{1-n} dn$:

$$\frac{\partial g(\theta,k,\ell)}{\partial\ell} = \frac{1}{(1-k)\ell}\left[\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)(1-\alpha\ell)}{1-n} dn - \ell\int_{\bar{n}}^{1} \frac{1}{n} dn - \int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)}{(1-n)}\left(1-\frac{(1-k)n}{\ell}\right) dn\right].$$

Since, from (32), we have that $\frac{\partial g(\theta,k,\ell)}{\partial k} = \frac{1}{(1-k)^2}\left[\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)(1-\alpha\ell)}{1-n} dn - \ell\int_{\bar{n}}^{1} \frac{1}{n} dn\right]$, we can write

$$\begin{aligned}\frac{\partial g(\theta,k,\ell)}{\partial\ell} &= \frac{(1-k)}{\ell}\frac{\partial g(\theta,k,\ell)}{\partial k} - \frac{1}{(1-k)\ell}\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)}{(1-n)}\left(1-\frac{(1-k)n}{\ell}\right) dn = \\ &= \frac{1}{\ell}\left[(1-k)\frac{\partial g(\theta,k,\ell)}{\partial k} - \frac{1}{(1-k)}\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)}{(1-n)}\left(1-\frac{(1-k)n}{\ell}\right) dn\right]. \quad (38)\end{aligned}$$

From (38), then, it is easy to see that when $k \leq \widetilde{k}(\ell)$ $\frac{\partial g(\theta,k,\ell)}{\partial\ell} < 0$, as $\frac{\partial g(\theta,k,\ell)}{\partial k} \leq 0$. This implies that $\frac{\partial\theta^*}{\partial\ell} > 0$ in the region below the curve $\widetilde{k}(\ell)$.

Consider now the range $\left(\widetilde{k}(\ell),\ k^{\max}(\ell)\right)$. We want to show that there are levels of bank capitalization $k \in \left(\widetilde{k}(\ell),\ k^{\max}(\ell)\right)$ for which increasing liquidity leads to a lower probability of liquidity-driven runs, i.e., $\frac{\partial\theta^*}{\partial\ell} < 0$. To do this, we need to show that there exist a region of k and ℓ, where the expression in the bracket in (33) is negative.

Rearrange the terms in the square bracket in (33) as follows:

$$-\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R(\theta^*)}{1-n}\left[n\frac{(1-k)}{\ell} - \alpha\ell\right] dn + \ell\int_{\bar{n}}^{1} \frac{1}{n} dn.$$

Using $Log\left(\Lambda\right) = -\int_{\widehat{n}(\theta^*)}^{\overline{n}} \frac{1}{1-n} dn$ and $\overline{n} = \frac{\ell}{(1-k)}$, the expression above simplifies to

$$-R\left(\theta^*\right)\alpha\ell Log\left(\Lambda\right) - \ell Log\left(\overline{n}\right) + \frac{R\left(\theta^*\right)\left(\overline{n} - \widehat{n}\left(\theta^*\right)\right)}{\overline{n}}\frac{(1-k)}{\ell} + \frac{R\left(\theta^*\right)Log\left(\Lambda\right)}{\overline{n}}\frac{(1-k)}{\ell}. \tag{39}$$

Since $\widehat{n}\left(\theta^*\right) = \overline{n}\frac{R(\theta^*)(1-\alpha\ell)-(1-k)r_2}{R(\theta^*)(1-\alpha\ell)-\ell r_2}$, we can rearrange the expression above as

$$R\left(\theta^*\right)Log\left(\Lambda\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right) - \ell Log\left(\overline{n}\right) + R\left(\theta^*\right)\frac{\left[(1-k)-\ell\right]r_2}{R\left(\theta^*\right)(1-\alpha\ell) - \ell r_2},$$

and further as

$$R\left(\theta^*\right)Log\left(\Lambda\right)\left(\frac{(1-k)}{\ell} - \alpha\ell\right) - \ell Log\left(\overline{n}\right) + R\left(\theta^*\right)\frac{\left[(1-k)-\ell\right]r_2}{R\left(\theta^*\right)(1-\alpha\ell) - \ell r_2}.$$

After a few manipulations, the expression above can be rearranged as

$$\begin{aligned} &\frac{R\left(\theta^*\right)}{\left[R\left(\theta^*\right)(1-\alpha\ell) - \ell r_2\right]}\left[Log\left(\Lambda\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)R\left(\theta^*\right)(1-\alpha\ell) - \ell r_2 Log\left(\Lambda\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)\right. \\ &\left. -\ell Log\left(\overline{n}\right)(1-\alpha\ell) + \frac{\ell^2 r_2 Log\left(\overline{n}\right)}{R\left(\theta^*\right)} + \left[(1-k)-\ell\right]r_2\right]. \end{aligned} \tag{40}$$

The sign of the expression above is determined by the sign of the terms in the square bracket, which, after a few manipulations, we can rearrange as follows

$$\begin{aligned} &\frac{\ell r_2}{R\left(\theta^*\right)}\left[\ell Log\left(\overline{n}\right) - R\left(\theta^*\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)Log\left(\Lambda\right)\right] + \left[(1-k)-\ell\right]r_2 \\ &-(1-\alpha\ell)\left[\ell Log\left(\overline{n}\right) - R\left(\theta^*\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)Log\left(\Lambda\right)\right], \end{aligned}$$

and further as

$$\left[R\left(\theta^*\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)Log\left(\Lambda\right) - \ell Log\left(\overline{n}\right)\right]\left(1 - \alpha\ell - \frac{\ell r_2}{R\left(\theta^*\right)}\right) + \left[(1-k)-\ell\right]r_2.$$

Multiply and divide the expression above by $R\left(\theta^*\right)\left[(1-k)-\ell\right]r_2$. It becomes

$$\frac{R\left(\theta^*\right)\left[(1-k)-\ell\right]r_2}{R\left(\theta^*\right)}\left[\left[R\left(\theta^*\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)Log\left(\Lambda\right) - \ell Log\left(\overline{n}\right)\right]\frac{R\left(\theta^*\right)(1-\alpha\ell) - \ell r_2}{R\left(\theta^*\right)\left[(1-k)-\ell\right]r_2} + 1\right]. \tag{41}$$

We want to show that $\left[R\left(\theta^*\right)\left(\frac{1}{\overline{n}} - \alpha\ell\right)Log\left(\Lambda\right) - \ell Log\left(\overline{n}\right)\right]\frac{R(\theta^*)(1-\alpha\ell)-\ell r_2}{R(\theta^*)[(1-k)-\ell]r_2} + 1$ can be negative for some $\{k, \ell\}$ between the curves $\widetilde{k}\left(\ell\right)$ and $k^{\max}\left(\ell\right)$. First, notice that $\frac{R(\theta^*)(1-\alpha\ell)-\ell r_2}{R(\theta^*)[(1-k)-\ell]r_2} > 1$

as long as $1-\alpha\ell > (1-k)\,r_2$ holds because $R(\theta^*) > r_2 \geq 1$ must hold to guarantee positive profits also for $\ell = 0$. Thus, we denote the curve $k^T(\ell)$ as the solution to

$$(1-k)\,r_2 = 1-\alpha\ell.$$

The curve $k^T(\ell)$ is increasing in the plane $\{k,\ell\}$, $k^T(0) = 1-\frac{1}{r_2}$ and $k^T(1) = 1-\frac{1}{r_2}+\frac{\alpha\ell}{r_2}$. It follows that for any pair $\{k,\ell\}$ above the curve $k^T(\ell)$ $\frac{R(\theta^*)(1-\alpha\ell)-\ell r_2}{R(\theta^*)[(1-k)-\ell]r_2} > 1$, while the opposite is true for pairs below the curve. Second, denote as $\widehat{k}(\ell)$, the pairs $\{k,\ell\}$ for which

$$R(\theta^*)\left(\frac{1}{\overline{n}}-\alpha\ell\right) Log\,(\Lambda) - \ell Log\,(\overline{n}) = -1.$$

Writing $1 = Log\,(e)$, we can rewrite the expression above as

$$Log\left(\Lambda^{R(\theta^*)\left(\frac{1}{\overline{n}}-\alpha\ell\right)}\right) - \ell Log\,(\overline{n}) = -Log(e),$$

thus obtaining

$$Log\left(\Lambda^{-R(\theta^*)\left(\frac{1}{\overline{n}}-\alpha\ell\right)}\right) = Log(\frac{e}{\overline{n}}),$$

and finally

$$k = 1-\ell e^{-\frac{1}{\ell}}\Lambda^{-\frac{R(\theta^*)\left(\frac{1}{\overline{n}}-\alpha\ell\right)}{\ell}}. \tag{42}$$

From (42), the $\underset{\ell^+\to 0}{Lim}\,\ell e^{-\frac{1}{\ell}}\Lambda^{-\frac{R(\theta^*)\left(\frac{1}{\overline{n}}-\alpha\ell\right)}{\ell}} = 0$, thus when ℓ approaches 0, $\widehat{k}$ approaches 1. Furthermore, $\widehat{k} = 0$ requires $\ell >> 0$. On the curve $\widehat{k}(\ell)$ $\frac{\partial\theta^*}{\partial\ell} < 0$ holds because the expression in (41) is negative. The curve $\widehat{k}(\ell)$ must lie below $k^{\max}(\ell)$ and above $\widetilde{k}(\ell)$ as $\widehat{k}(\ell)$ above $k^{\max}(\ell)$ and $\widehat{k}(\ell)$ below $\widetilde{k}(\ell)$ would contradict the result that $\frac{\partial\theta^*}{\partial\ell} > 0$ below the curve $\widetilde{k}(\ell)$ and above $k^{\max}(\ell)$.

Given that $\frac{\partial\theta^*}{\partial\ell} > 0$ for pairs $\{k,\ell\}$ below the curve $\widetilde{k}(\ell)$ and above the curve $k^{\max}(\ell)$, by continuity, there exist two thresholds $\underline{k}(\ell) \in \left(\widetilde{k}(\ell), \widehat{k}(\ell)\right)$ and $\overline{k}(\ell) \in \left(\widehat{k}(\ell), k^{\max}(\ell)\right)$, such that $\frac{\partial\theta^*}{\partial\ell} > 0$ for pairs $\{k,\ell\}$ between the curves $\widetilde{k}(\ell)$ and $\underline{k}(\ell)$, and $\overline{k}(\ell)$ and $k^{\max}(\ell)$, while $\frac{\partial\theta^*}{\partial\ell} < 0$ for pairs $\{k,\ell\}$ between the curves $\overline{k}(\ell)$ and $\underline{k}(\ell)$.[9] Thus, the proposition follows. □

Proof of Proposition 2: The proof proceeds in steps. First, we characterize the equilibrium choice of k, ℓ and r_2. Second, we show that the equilibrium k and ℓ are consistent with $\frac{\partial\theta^*}{\partial k} < 0$ and $\frac{\partial\theta^*}{\partial\ell} < 0$. Finally, we show that in equilibrium banks choose k and ℓ in such a way

[9] Note that $\frac{R(\theta^*)(1-\alpha\ell)-\ell r_2}{R(\theta^*)[(1-k)-\ell]r_2} > 1$ is a sufficient condition for $\frac{\partial\theta^*}{\partial\ell} < 0$. This implies that for pairs $\{k,\ell\}$ below the curve $k^T(\ell)$, it could still be that $\frac{\partial\theta^*}{\partial\ell} < 0$. In that case, it holds that $\frac{\partial\theta^*}{\partial\ell} < 0$ for any pair $\{k,\ell\}$ between the curves $\underline{k}(\ell)$ and $\overline{k}(\ell)$.

that $(1-k) > \ell$ holds in equilibrium so that liquidity crises occur.

Before starting solving the bank's problem, it is important to notice that the interest rate r_2 affects the threshold θ^* and it is chosen at date 0 from the debt holder's participation constraint, thus anticipating the withdrawal threshold θ^*. Differentiating the LHS of (11) with respect to θ^*, we obtain

$$-\left[r_2 - \frac{\ell}{1-k}\right] + \int_{\theta^*}^{1} \frac{dr_2}{d\theta^*} d\theta, \tag{43}$$

where $\frac{dr_2}{d\theta^*}$ can be computed using the implicit function theorem from (9) and it is then equal to

$$-\frac{\int_{\widehat{n}(\theta^*)}^{\bar{n}} \frac{R'(\theta^*)(1-\alpha\ell)\left[1-\frac{(1-k)n}{\ell}\right]}{(1-k)(1-n)} dn}{\int_0^{\widehat{n}(\theta^*)} dn} < 0.$$

This implies that the expression in (43) is negative and so that each pair $\{k, \ell\}$ implements only one θ^*.

Now we move on to solve bank's optimal choice of its capital structure and portfolio liquidity. The conditions (14) and (15) in the proposition are obtained by substituting r_2 from (11) into (10) and differentiating it with respect to k and ℓ.

Concerning the proof that the bank's choice is always consistent with $\frac{\partial\theta^*}{\partial k} < 0$ and $\frac{\partial\theta^*}{\partial \ell} < 0$, first notice that a higher θ^* leads to lower bank expected profit as a run becomes more likely and, from (11), when there is no for $\theta > \theta^*$, a higher run probability translates into a higher r_2. Thus, we need to show that the effect of a change in k and ℓ on the threshold θ^* is positive when $\frac{\partial\theta^*}{\partial k} > 0$ and $\frac{\partial\theta^*}{\partial \ell} > 0$, even accounting for the indirect effect of k and ℓ on θ^* via r_2.

We can compute the total effect of k on θ^* $\frac{d\theta^*}{dk}$ as follows. Implicitly differentiating (11) with respect to k, we obtain

$$\frac{d\theta^*}{dk} = -\frac{\int_0^{\theta^*} \frac{\ell}{(1-k)^2} d\theta + \int_{\theta^*}^{1} \frac{dr_2}{dk} d\theta}{-\left[r_2 - \frac{\ell}{(1-k)}\right] + \int_{\theta^*}^{1} \frac{dr_2}{d\theta^*} d\theta},$$

where $\frac{\partial r_2}{\partial \theta^*} < 0$ and $\frac{\partial r_2}{\partial k}$ is obtained by implicitly differentiating (9) and is equal to

$$\frac{dr_2}{dk} = -\frac{\frac{\partial g(\theta^*,k,\ell)}{\partial k}}{\frac{\partial g(\theta^*,k,\ell)}{\partial r_2}}.$$

Given that $\frac{\partial g(\theta^*,k,\ell)}{\partial r_2} > 0$, as long as $\frac{\partial g(\theta^*,k,\ell)}{\partial k} < 0$, $\frac{dr_2}{dk} > 0$ and $\frac{d\theta^*}{dk} > 0$ since $-\left[r_2 - \frac{\ell}{(1-k)}\right] + \int_{\theta^*}^{1} \frac{\partial r_2}{\partial \theta^*} d\theta < 0$. As shown in the proof of Corollary 1, $\frac{\partial g(\theta^*,k,\ell)}{\partial k} < 0$ for pairs $\{k, \ell\}$ below the curve $\widetilde{k}(\ell)$. Following the same steps to compute $\frac{d\theta^*}{d\ell}$, we have that

$$\frac{d\theta^*}{d\ell} = -\frac{\int_0^{\theta^*} \frac{1}{(1-k)} d\theta + \int_{\theta^*}^1 \frac{dr_2}{d\ell} d\theta}{-\left[r_2 - \frac{\ell}{(1-k)}\right] + \int_{\theta^*}^1 \frac{dr_2}{d\theta^*} d\theta},$$

with

$$\frac{\partial r_2}{\partial \ell} = -\frac{\frac{\partial g(\theta^*,k,\ell)}{\partial \ell}}{\frac{\partial g(\theta^*,k,\ell)}{\partial r_2}}.$$

The derivative $\frac{dr_2}{d\ell}$ and, in turn, $\frac{d\theta^*}{d\ell}$ are positive when $\frac{\partial g(\theta^*,k,\ell)}{\partial \ell} < 0$. As shown in the proof of Corollary 1, this is the case for any pair $\{k,\ell\}$ below the curve $\underline{k}(\ell)$ and above the curve $\overline{k}(\ell)$. Thus, the bank would only choose a pair $\{k,\ell\}$ in the region bounded by the curves $\underline{k}(\ell)$ and $\overline{k}(\ell)$.

To complete the proof, we need to show that the equilibrium k and ℓ satisfy $(1-k) > \ell$. To see this, we rearrange the first order conditions for k and ℓ:

$$\begin{array}{c} -\frac{\partial\theta^*}{\partial k}\left[R(\theta^*)(1-\alpha\ell) - (1-k)r_2\right] + \int_{\theta^*}^1 r_2 d\theta - \rho \\ -\frac{dr_2}{dk}\left\{\int_{\theta^*}^1 (1-k)\, d\theta + \frac{\partial\theta^*}{\partial r_2}\left[R(\theta^*)(1-\alpha\ell) - (1-k)r_2\right]\right\} = 0 \end{array}, \tag{44}$$

and

$$\begin{array}{c} -\frac{\partial\theta^*}{\partial \ell}\left[R(\theta^*)(1-\alpha\ell) - (1-k)r_2\right] - \int_{\theta^*}^1 \alpha R(\theta)\, d\theta \\ -\frac{dr_2}{d\ell}\left\{\int_{\theta^*}^1 (1-k)\, d\theta + \frac{\partial\theta^*}{\partial r_2}\left[R(\theta^*)(1-\alpha\ell) - (1-k)r_2\right]\right\} = 0 \end{array}. \tag{45}$$

Assume that a bank sets $\ell = (1-k)$. From (11), it follows immediately that $r_2^* = 1$. Then, the expression (44) simplifies to

$$\int_{\underline{\theta}}^1 d\theta - \rho - \left.\frac{dr_2}{dk}\right|_{(1-k)=\ell} \int_{\underline{\theta}}^1 (1-k)\, d\theta.$$

The derivative $\frac{dr_2}{dk}$ can be computed using the implicit function theorem on (11) and is, then equal to

$$\left.\frac{dr_2}{dk}\right|_{(1-k)=\ell} = -\frac{\int_0^{\underline{\theta}} \frac{\ell}{(1-k)^2} d\theta}{\int_{\underline{\theta}}^1 d\theta} < 0,$$

which implies that the expression for (44) evaluated at $(1-k) = \ell$ can be rearranged as

$$\int_{\underline{\theta}}^1 d\theta - \rho + \int_0^{\underline{\theta}} d\theta = -(\rho - 1) < 0.$$

The fact that (44) is negative when evaluated at $(1-k)=\ell$ implies that the bank will always choose a level of k so that the inequality $(1-k)>\ell$ holds in equilibrium. This, in turn, implies that $r_2^*>1$ for the (11) to be satisfied. Thus, the proposition follows. □

Proof of Proposition 3: The regulator maximizes (19) subject to the constraint that the bank's profits in (10) are non-negative, $r_2=\arg\max \Pi^B$, which implies solving the depositors' participation constraint in (11) with equality.

From (11), we can obtain

$$\int_{\theta^*}^{1}(1-k)\,r_2 d\theta=(1-k)-\int_{0}^{\theta^*}\ell d\theta.$$

Then, substituting it into (10), we can rearrange the expression for bank profits as follows

$$\Pi^B=\int_{0}^{\theta^*}\ell d\theta+\int_{\theta^*}^{1}R\left(\theta\right)\left(1-\alpha\ell\right)d\theta-(1-k)-\rho k.$$

The Lagrangian for the regulator's problem is given by

$$\mathcal{L}=\left[\int_{0}^{\theta^*}\ell d\theta+\int_{\theta^*}^{1}R\left(\theta\right)\left(1-\alpha\ell\right)d\theta\right]\left(1+\lambda_1\right)-\lambda_1\rho k-\lambda_1\left(1-k\right),$$

where λ_1 is the Lagrangian multipliers for the non-negativity condition on Π^B. The Kuhn-Tucker conditions are as follows:

$$\frac{\partial\mathcal{L}}{\partial k}=-\left[\frac{\partial\theta^*}{\partial k}+\frac{\partial\theta^*}{\partial r_2}\frac{\partial r_2}{\partial k}\right]\left[R\left(\theta^*\right)\left(1-\alpha\ell\right)-\ell\right]\left(1+\lambda_1\right)-\lambda_1(\rho-1)=0; \tag{46}$$

$$\frac{\partial\mathcal{L}}{\partial \ell}=-\left[\frac{\partial\theta^*}{\partial \ell}+\frac{\partial\theta^*}{\partial r_2}\frac{\partial r_2}{\partial \ell}\right]\left[R\left(\theta^*\right)\left(1-\alpha\ell\right)-\ell\right]\left(1+\lambda_1\right)+\left[\int_{0}^{\theta^*}d\theta-\int_{\theta^*}^{1}\alpha R\left(\theta\right)d\theta\right]\left(1+\lambda_1\right)=0; \tag{47}$$

$$\lambda_1\geq 0;\ \Pi^B\geq 0 \tag{48}$$

$$\lambda_1\left[\int_{0}^{\theta^*}\ell d\theta+\int_{\theta^*}^{1}R\left(\theta\right)\left(1-\alpha\ell\right)d\theta-(1-k)-\rho k\right]=0. \tag{49}$$

From (47) it can be easily seen that for a given k the bank's and regulator's choice of ℓ are the same as the expression in (47) is the same as (15). This implies that no liquidity regulation is needed: The regulator only sets capital requirements and the bank chooses the level of liquidity given the level of capitalization k chosen by the regulator. In what follows, we then consider that liquidity ℓ is chosen by the bank as the solution to (15).

Suppose that $\lambda_1 = 0$. This means that from (49), $\Pi^B = \int_0^{\theta^*} \ell d\theta + \int_{\theta^*}^1 R(\theta)(1-\alpha\ell)\, d\theta - (1-k) - \rho k > 0$ in the candidate maximum. When $\lambda_1 = 0$, we can rewrite (46) and (15) as, respectively,

$$-\frac{\partial\theta^*}{\partial k}\left[R(\theta^*)(1-\alpha\ell) - \ell\right] = 0; \tag{50}$$

$$-\frac{\partial\theta^*}{\partial\ell}\left[R(\theta^*)(1-\alpha\ell) - \ell\right] + \left[\int_0^{\theta^*} d\theta - \int_{\theta^*}^1 \alpha R(\theta)\, d\theta\right] = 0; \tag{51}$$

It follows from (50) that $k^R = 1 - \ell^R$ must hold, while ℓ^R corresponds to the solution to (51) which becomes

$$\int_0^{\theta^E} d\theta - \int_{\theta^E}^1 \alpha R(\theta)\, d\theta = 0, \tag{52}$$

since, when $k^R = 1 - \ell^R$, $\theta^* \to \underline{\theta} = \theta^E$ and so $R\left(\theta^E\right)(1-\alpha\ell) - \ell = 0$. In this case, the social welfare can be rewritten as

$$\int_0^{\theta^E} \left(1 - k^R\right) d\theta + \int_{\theta^E}^1 R(\theta)\left(1 - \alpha(1-k^R)\right) d\theta,$$

and the bank's profit as

$$\int_0^{\theta^E} \left(1 - k^R\right) d\theta + \int_{\theta^E}^1 R(\theta)\left(1 - \alpha(1-k^R)\right) d\theta - \rho k^R - (1 - k^R). \tag{53}$$

By choosing $k^R = 1$, the regulator eliminates all runs (i.e., $\theta^E = 0$) and so the need for costly liquidity, thus maximizing social welfare. From (52), it is easy to see that the bank sets $\ell^R = 0$. When $\ell^R = 0$ and $k^R = 1$, $\theta^E = 0$ and the expression in (53) becomes

$$\int_0^1 R(\theta)\, d\theta - \rho. \tag{54}$$

Denote as $\overline{\rho}$ the solution to the expression in (54) equal to zero. For any $\rho > \overline{\rho}$, $\Pi^B < 0$, while $\Pi^B \geq 0$ for $\rho \leq \overline{\rho}$. This implies that, when $\rho \leq \overline{\rho}$, the regulator's choice is $k^R = 1$ and $\ell^R = 0$ solving (52). From (11), it is easy to see that in this case $r_2^R = 1$.

When $\rho > \overline{\rho}$, $k^R = 1$ and $\ell^R = 0$ cannot be a solution, as it violates the non-negativity constraint on the bank's profit. We check here whether a solution featuring $k^R = 1 - \ell^R$ is feasible in that is consistent with the bank accruing non-negative profits. From (53), using

$k^R = 1 - \ell^R$, it is easy to see that Π^B increases in ℓ, as

$$\begin{aligned}\frac{\partial \Pi^B}{\partial \ell} &= \int_0^{\theta^E} d\theta - \int_{\theta^E}^1 \alpha R(\theta)\, d\theta + \rho - 1 = \\ &= -\int_{\theta^E}^1 1 d\theta - \int_{\theta^E}^1 \alpha R(\theta)\, d\theta + \rho > 0,\end{aligned}$$

given that $\int_0^1 R(\theta)(1-\alpha)\, d\theta > 1$ must hold in order for the bank to invest in the risky portfolio, it follows that $\int_0^1 R(\theta)\, d\theta > 1 + \int_0^1 R(\theta)\, \alpha d\theta$, which is, in turn, greater than $\int_{\theta^E}^1 1 d\theta + \int_{\theta^E}^1 \alpha R(\theta)$. Thus, $\int_{\theta^E}^1 1 d\theta + \int_{\theta^E}^1 \alpha R(\theta) < \int_0^1 R(\theta)(1-\alpha)\, d\theta < \int_0^1 R(\theta)\, d\theta \leq \rho$ for any $\rho > \overline{\rho}$. Furthermore, Π^B decreases with both ρ and α since

$$\frac{\partial \Pi^B}{\partial \rho} = -\rho < 0,$$

and

$$\frac{\partial \Pi^B}{\partial \alpha} = -\int_{\theta^E}^1 R(\theta)\, \ell d\theta < 0.$$

This implies that, when both α and ρ increase, a higher ℓ is required for the bank profit to be non-negative. For a given ℓ, we can denote as $\widehat{\rho}(\alpha)$ as the combinations of ρ and α for which $\Pi^B = 0$. The curve $\widehat{\rho}(\alpha)$ is downward sloping in the plane $\{\alpha, \rho\}$ and takes value $\overline{\rho}$ when $\ell = 0$, while it corresponds to a vertical line when $\ell = 1$. For any pair $\{\alpha, \rho\}$ below $\widehat{\rho}(\alpha)$, $\Pi^B > 0$, while $\Pi^B < 0$ above the curve $\widehat{\rho}(\alpha)$. Thus, for pairs $\{\alpha, \rho\}$ above the curve $\widehat{\rho}(\alpha)$, the equilibrium features $\lambda_1 > 0$ and $k^R < 1 - \ell^R < 1$.

For pairs $\{\alpha, \rho\}$ below the curve $\widehat{\rho}(\alpha)$, there are two candidate equilibria for the regulator's problem. One features only efficient runs and is characterized by $k^R = 1 - \ell^R < 1$ and solving $\Pi^B = 0$. The other, instead, features $k^R < 1 - \ell^R < 1$, $\ell^R > 0$, $\lambda_1 > 0$ with k^R, ℓ^R and λ_1 corresponding to the solution of (46) and (15) and

$$\Pi^B = \int_0^{\theta^*} \ell d\theta + \int_{\theta^*}^1 R(\theta)(1 - \alpha\ell)\, d\theta - (1-k) - \rho k = 0.$$

Comparing social welfare in the two equilibria, it is easy to see that the equilibrium featuring the highest welfare is the one characterized by the highest level of capital, since, using (19), we can rewrite

$$SW = 1 + (\rho - 1)\, k$$

This follows directly from the fact that, in both equilibria, $\Pi^B = 0$, debt holders receive 1 in expectation and $\rho > 1$. The candidate equilibrium featuring only efficient runs features $k^R = 1 - \ell^R$ and requires a high ℓ^R when ρ and α are large, while in the equilibrium in which liquidity crises occur an increase in ℓ^R does not necessarily requires k^R to decrease by one (i.e., $\frac{\partial k^R}{\partial \ell^R} \neq 1$). This implies that the equilibrium features no liquidity crises only when ρ and α are sufficiently low. We then denote as $\widetilde{\rho}(\alpha)$ the curve below which eliminating inefficient liquidity crises is optimal. The curve $\widetilde{\rho}(\alpha)$ corresponds to the pair $\{\alpha, \rho\}$ for which the two candidate equilibria entail the same k and so the same welfare.

When $\rho > \widetilde{\rho}(\alpha)$, the equilibrium features $k^R < 1 - \ell^R$, ℓ^R and λ_1 corresponding to the solution of (46) and (15) and

$$\Pi^B = \int_0^{\theta^*} \ell d\theta + \int_{\theta^*}^1 R(\theta)(1 - \alpha\ell) d\theta - (1 - k) - \rho k = 0.$$

The solution $\{k^R, \ell^R\}$ lies in the region where $\frac{\partial \theta^*}{\partial k} < 0$, as, otherwise, (46) does not hold. Given that $k^R < 1 - \ell^R$ in equilibrium, it follows from (11) that $r_2 > 1$. Thus, the proposition follows. $\square$

Proof of Proposition 4: The proof proceeds in steps. First, we prove that in any bank debt holders behave according to a threshold strategy when assuming that debt holders in other banks also behave according to a threshold strategy. Second, we characterize the equilibrium thresholds. Finally, we show that they are unique.

When debt holders at any bank behave according to a threshold strategy s_g^*, the proportion of debt holders in bank g withdrawing at date 1 is equal to the probability of receiving a signal below s_g^*. We denote it as $n_g = n_g(\theta, s_g^*)$ and it is still given by (29).

A debt holder's utility differential is similar to the one in (5) with the difference that the bank liquidates $\frac{(1-k_g)n_g}{\chi(Q)}$ units of the portfolio to meet the $(1 - k_g)\, n_g$ early withdrawals rather than $\frac{(1-k_g)n_g}{\ell_g}$ as in the baseline model. This implies that a debt holder's utility differential is also a function of $n_{(-g)}$ that is of the proportion of other debt holders withdrawing at date 1 in all other banks.

The proof that a debt holder in bank g behave according to a threshold strategy when all other debt holders in the economy also behave according to a threshold strategy follows the same steps as in the proof of Proposition 1. Thus, denote as $\Delta(s_i, \dot{n}(\theta), \dot{\mathbf{n}}_{(-g)}(\theta))$ the expected difference in utility between withdrawing at date 2 and at date 1 of debt holder i in bank g when he holds beliefs $\dot{n}(\theta)$ and $\dot{\mathbf{n}}_{(-g)}(\theta)$ regarding the number of depositors running in his own bank and in all the other banks in the economy. The function $\Delta(s_i, \dot{n}(\theta), \dot{\mathbf{n}}_{(-g)}(\theta))$ is given by

$$\Delta(s_{ig}, \dot{n}(\theta), \dot{n}_{(-g)}(\theta)) = \frac{1}{2\varepsilon} \int_{s_i - \varepsilon}^{s_i + \varepsilon} E_n \left[v(\theta, \dot{n}(\theta), \dot{\mathbf{n}}_{(-g)}(\theta)) \right] d\theta.$$

Since for any realization of θ, the proportion of depositors running is deterministic, we can write $n_g(\theta)$ and $\mathbf{n}_{(-g)}(\theta)$ instead of $\dot{n}_g(\theta)$ and $\dot{\mathbf{n}}_{(-g)}(\theta)$ and the function $\Delta(s_i, \dot{n}(\theta), \dot{\mathbf{n}}_{(-g)}(\theta))$

simplifies to

$$\Delta(s_i, \dot{n}(\theta), \dot{\mathbf{n}}_{(-g)}(\theta)) = \frac{1}{2\varepsilon} \int_{s_i-\varepsilon}^{s_i+\varepsilon} v(\theta, n(\theta), \mathbf{n}_{(-g)}(\theta)) d\theta.$$

Notice that when all debt holders behave according to the same threshold strategy s^*, $\dot{n}(\theta) = n(s^*, \theta)$ as defined in (29). As in the proof of Proposition 1, for a unique threshold signal to exist we need to show that the utility differential of a debt holder in bank g is decreasing in n and increasing in θ. Regarding the former, the $v(.)$ is as in the baseline model, thus still exhibiting the property of one-sided strategic complementarity. Regarding the latter, unlike the baseline model, we also need to account the effect that θ has on $v(\theta, n(\theta), \mathbf{n}_{(-g)}(\theta))$ through its effect on the proportion of debt holders running in other banks $\mathbf{n}_{(-g)}(\theta)$. Specifically, the effect of θ on the $v(.)$ is given by

$$\begin{aligned} &\int_{\widehat{n}(\theta)}^{\overline{n}} \frac{R'(\theta)(1-\alpha\ell_g)\left[1-\frac{n_g(1-k_g)}{\chi(Q)}\right]}{(1-k_g)(1-n_g)} dn_g \\ &+ \left[\int_{\widehat{n}(\theta)}^{\overline{n}} \frac{R(\theta)(1-\alpha\ell_g)\frac{n_g(1-k_g)}{\chi(Q)}}{(1-k_g)(1-n_g)} dn_g - \int_{\overline{n}}^{1} \frac{\chi(Q)}{(1-k_g)\, n_g} dn_g\right] \frac{\chi'(Q,w)}{\chi(Q)}, \end{aligned} \tag{55}$$

since the derivatives of the extreme of the integrals cancel out and given (29) and $\frac{\partial Q}{\partial n_g} > 0$ it is easy to see that $\chi'(Q) > 0$ since as θ increases the proportion of debt holders running in all other banks decreases for all $g = 1, ...G$. The expression in (55) is positive for $R'(\theta)$ sufficiently large.

The analysis above implies that, even accounting for the effect of θ on the proportion of debt holders running in all other banks, the function $v(.)$ exhibits the same properties as in the baseline model. Thus, the rest of the proof goes through and all debt holders in bank g withdraw at date 1 if they receive a signal below s_g^* and roll over otherwise when they expect debt holders in the other banks also to behave according to a threshold strategy. The condition (22) in the proposition represents a debt holder's indifference condition between rolling over the debt and withdrawing at date 1 and it is obtained by substituting $\theta = s_g^* + \varepsilon - 2\varepsilon n_g$ into the expression for the proportion of early withdrawing debt holders $n_{(-g)}$ for all banks other than g as given in (29). The equilibrium corresponds to the vector of threshold signals $\mathbf{s}^*$ solving the system of G indifference condition as the one given in (22).

To complete the proof we need to show that the system of G indifference conditions has a unique solution. Denote as $f_g\left(s_g^*, \mathbf{s}_{(-g)}^*\right) = 0$ each indifference condition. We can rearrange the system in a matrix form as $\mathbf{A}\mathbf{s}^* = \mathbf{b}$, with $\mathbf{b} \neq \mathbf{0}$.

The matrix of the coefficients $\mathbf{A}$ is equal to

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1(.)}{\partial s_1^*} & \frac{\partial f_1(.)}{\partial s_2^*} & \cdots & \frac{\partial f_1(.)}{\partial s_G^*} \\ . & & & . \\ . & \frac{\partial f_2(.)}{\partial s_2^*} & \cdots & . \\ . & & & . \\ \frac{\partial f_G(.)}{\partial s_1^*} & \cdots & \cdots & \frac{\partial f_G(.)}{\partial s_G^*} \end{bmatrix},$$

where the terms on the diagonal capture the effect of the threshold signal s_g^* on the indifference condition of a debt holder in bank g (i.e., $\frac{\partial f_g(.)}{\partial s_g^*}$), while all other terms are the effect of the threshold signal of debt holders in a bank other than g on the indifference condition of a debt holder in bank g (i.e., $\frac{\partial f_g(.)}{\partial s_{(-g)}^*}$). From (22), it is easy to see that $\frac{\partial f_g(.)}{\partial s_g^*} > 0$ while $\frac{\partial f_g(.)}{\partial s_{(-g)}^*} < 0$ and $\left|\frac{\partial f_g(.)}{\partial s_g^*}\right| > \left|\frac{\partial f_g(.)}{\partial s_{(-g)}^*}\right|$, as the latter also includes the direct effect that s_g^* has on date 2 per unit return $R\left(s_g^* + \varepsilon - 2\varepsilon n_g\right)(1-\alpha\ell)$ on top of the effect of the signals on $\chi(.)$. Furthermore, given the bank are symmetric, in equilibrium they choose the same k_g, ℓ_g and r_{2g}. This implies that $\frac{\partial f_g(.)}{\partial s_{(-g)}^*}$ is the same for all g and $\frac{\partial f_g(.)}{\partial s_g^*} = \frac{\partial f_{(-g)}(.)}{\partial s_{(-g)}^*}$. Then, it follows that the determinant of matrix $\mathbf{A}$ is equal to

$$\left(\frac{\partial f_g(.)}{\partial s_g^*} - \frac{\partial f_g(.)}{\partial s_{(-g)}^*}\right)^{(G-1)} \left(\frac{\partial f_g(.)}{\partial s_g^*} + (G-1)\frac{\partial f_{(-g)}(.)}{\partial s_g^*}\right) \neq 0$$

and the system of G indifference conditions has a unique solution, which we denote as the vector $\mathbf{s}_G^*$. Thus, the proposition follows. □

Proof of Proposition 5: The proof follows Goldstein (2005) and it is done for $\varepsilon \to 0$, so that $s_g^* \to \theta_g^*$ given that $\theta = s_g^* + \varepsilon - 2\varepsilon n_g$. The arguments in his proof establish that there is a unique threshold of fundamental θ, which we denote as θ_G^*, below which debt holders at all bank withdraw at date 1 and roll over otherwise.

The proof hinges on the characterization of the equilibrium thresholds in the case where debt holders in a bank g have extreme beliefs about the actions of debt holders in the other banks. Denote as $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{1}\right)$ and $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{0}\right)$ debt holders' equilibrium threshold in the case they expect that no investors roll over and all investors roll over, respectively, in all other $-g$ banks in the economy. As banks are symmetric, $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{1}\right)$ and $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{0}\right)$ are the same for all banks. These thresholds under extreme beliefs can be computed following the same steps illustrated in Proposition 4 but fixing the proportion of debt holders running in other banks. Notice that the threshold $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{0}\right)$ is the same as θ^* characterized in Proposition 1 since if only one bank sells asset in the market no fire sale occurs and $\chi(Q) = \ell_g$. Since threshold characterized in Proposition 4 are computed for $0 \leq n_{(-g)} \leq 1$ and the actions of debt holders in different banks are strategic complements, it follows that equilibrium thresholds θ_g^* lies in the range $(\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{0}\right), \theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{1}\right)]$ that is it is strictly larger than $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{0}\right) \equiv \theta^*$.

To complete the proof we need to show that all θ_g^* converges to the same value θ_G^*. Assume by contradiction that $\theta_g^* < \theta_{(-g)}^*$. Then, a debt holder i in bank g receiving the signal $s_i = \theta_g^*$ is indifferent between running and rolling over and believes that debt holders in all other banks in the economy withdraw at date 1. Thus, θ_g^* would converge to $\theta_g^*\left(\mathbf{n}_{(-g)} = \mathbf{1}\right) > \theta_{(-g)}^*$. A similar argument rules out the possibility that $\theta_g^* > \theta_{(-g)}^*$. Thus, in equilibrium it must be that $\theta_g^* = \theta_{(-g)}^*$ and we denote it as $\overset{*}{\theta}_G$ and the proposition follows. $\square$

Proof of Proposition 6: The proof is analogous to that of Proposition 2. The conditions (25) and (26) in the proposition are obtained by substituting r_2 from (23) into (10) and differentiating it with respect to k and ℓ. Following the same steps as in the proof of Proposition 2, it can be shown that when banks choose $1 - k = \underline{\chi}(\boldsymbol{\ell})$, $\theta_G^* \to \underline{\theta}$, which is still given by (4). Furthermore, from (23), $r_2 = 1$ and the bracket $[R(\theta_G^*)(1 - \alpha\ell) - (1 - k)] = 0$ and so the expression in (25) becomes negative. This implies that $1 - k_G^B > \underline{\chi}(\boldsymbol{\ell})$ holds and so the proposition follows. $\square$

Proof of Proposition 7: The Lagrangian for the regulator's problem is given by

$$\mathcal{L} = \left[\int_0^{\theta_G^*} \underline{\chi}(\ell)\, d\theta + \int_{\theta_G^*}^1 R(\theta)(1 - \alpha\ell)\, d\theta\right](1 + \lambda_1) - \lambda_1 \rho k - \lambda_1 (1 - k) + \lambda_2 \left(\ell - \underline{\chi}(\ell)\right).$$

It only differs from that in the proof of Proposition 3 because the relevant run threshold is θ_G^* instead of θ^* and the regulator faces the additional constraint that $\underline{\chi}(\ell) \leq \ell$ so that the term $\lambda_2\left(\ell - \underline{\chi}(\ell)\right)$ also appears in the expression. The Kuhn-Tucker conditions are as (46)-(49) in the proof of Proposition 3 with two differences. First, condition (47) features the extra term $+\lambda_2\left[1 - \frac{\partial \underline{\chi}(\ell)}{\partial \ell}\right]$ and $\int_0^{\theta_G^*} \frac{\partial \underline{\chi}(\ell)}{\partial \ell} d\theta$ instead of $\int_0^{\theta^*} 1 d\theta$. Second, the conditions $\lambda_2 \geq 0$ and $\lambda_2\left(\ell - \underline{\chi}(\ell)\right) = 0$ must be added to the set of conditions.

Based on the same arguments as in the proof of Proposition 3, it holds that the regulator chooses $k^R = 1$, $\ell^R = 0$ as long as it is feasible, i.e., for $\rho \leq \overline{\rho}$.

Consider now the case $\rho > \overline{\rho}$. We start by characterizing the set of candidate equilibria. Then, we move on to identify the regions in the plane $\{\alpha, \rho\}$ in which each of them emerges as the solution to the regulator's maximization problem.

Suppose first that $\lambda_1 = 0$ and $\lambda_2 > 0$ so that $\underline{\chi}(\ell) = \ell$. This implies that in equilibrium $\ell \geq \widehat{\ell}$ must hold. From (46) we obtain $k^R = 1 - \ell^R$, which in turn implies that $\theta^* \to \underline{\theta} = \theta^E$, while (47) becomes

$$\int_0^{\theta^E} d\theta - \int_{\theta^E}^1 \alpha R(\theta)\, d\theta = 0,$$

since $\frac{\partial \underline{\chi}(\ell)}{\partial \ell}\Big|_{\ell=\widehat{\ell}} = 1$. Denote as $\widehat{\rho}(\alpha)$ the pairs $\{\alpha, \rho\}$ for which $\Pi^B\left(1-\widehat{\ell}, \widehat{\ell}\right) = 0$. As shown in the proof of Proposition 3, banks' profit evaluated at $k = 1-\ell$ increases with ℓ and decreases with α and ρ. Thus, the solution $k^R = 1-\widehat{\ell}$ and $\ell^R = \widehat{\ell}$ is feasible for pairs $\{\alpha, \rho\}$ below the curve $\widehat{\rho}(\alpha)$, while above the curve $\widehat{\rho}(\alpha)$, $\ell > \widehat{\ell}$ is required for the banks' profit to be non-negative.

Suppose now that $\lambda_1 = 0$ and $\lambda_2 = 0$, then $\underline{\chi}(\ell) < \ell$, or equivalently $\ell < \widehat{\ell}$, holds. From (46) we obtain $k^R = 1 - \underline{\chi}(\ell)$, which in turn implies that $\theta^* \to \underline{\theta} < \theta^E$, while from (47) we obtain

$$\int_0^{\underline{\theta}} \frac{\partial \underline{\chi}(\ell)}{\partial \ell} d\theta - \int_{\underline{\theta}}^1 \alpha R(\theta)\, d\theta = 0 \tag{56}$$

Since both (56) and $\Pi^B(1-\underline{\chi}(\ell), \ell)$ are increasing in ℓ, the regulator chooses ℓ^R as the solution to $\Pi^B\left(k^R, \ell^R\right) = 0$. As in the previous case, it is easy to see that $\Pi^B\left(1-\underline{\chi}(\ell), \ell\right)$ is decreasing in both ρ and α and increasing in ℓ. Thus, we denote as $\widehat{\widehat{\rho}}(\alpha)$ the pairs $\{\alpha, \rho\}$ for which $\Pi^B\left(1-\underline{\chi}(\ell), \ell\right) = 0$, so that the solution $k^R = 1-\underline{\chi}\left(\ell^R\right)$ and $\ell^R < \widehat{\ell}$ as given by the solution to $\Pi^B\left(1-\underline{\chi}\left(\ell^R\right), \ell^R\right) = 0$ is only feasible for pairs $\{\alpha, \rho\}$ in the region below the curve $\widehat{\widehat{\rho}}(\alpha)$.

Finally, suppose $\lambda_1 > 0$, then $k^R < 1-\ell^R$ and the relevant crisis threshold is θ^*. Specifically, the candidate equilibrium features λ_1, k^R and ℓ^R solving $\Pi^B\left(k^R, \ell^R\right) = 0$ and

$$-\left[\frac{\partial \theta_G^*}{\partial k} + \frac{\partial \theta_G^*}{\partial r_2}\frac{\partial r_2}{\partial k}\right]\left[R(\theta^*)(1-\alpha\ell) - \underline{\chi}(\ell)\right](1+\lambda_1) - \lambda_1(\rho-1) = 0,$$

and

$$-\left[\frac{\partial \theta_G^*}{\partial \ell} + \frac{\partial \theta_G^*}{\partial r_2}\frac{\partial r_2}{\partial \ell}\right]\left[R(\theta^*)(1-\alpha\ell) - \underline{\chi}(\ell)\right] + \int_0^{\theta_G^*} \frac{\partial \underline{\chi}(\ell)}{\partial \ell} d\theta - \int_{\theta_G^*}^1 R(\theta)\alpha d\theta = 0.$$

Given the three candidate equilibria described above, to determine which one emerges as the equilibrium, we need to consider different ranges of values for α and ρ. In particular, it is convenient to distinguish two cases depending on whether the curve $\widehat{\rho}(\alpha)$ lies below the curve $\widetilde{\rho}(\alpha)$, which we defined in the proof of Proposition 3.

When the curve $\widehat{\rho}(\alpha)$ lies below the curve $\widetilde{\rho}(\alpha)$, using the same arguments as in the proof of Proposition 3, the regulator optimally chooses $k^R = 1-\ell^R$ for pairs $\{\alpha, \rho\}$ in the region between the curves $\widehat{\rho}(\alpha)$ and $\widetilde{\rho}(\alpha)$, while for pairs $\{\alpha, \rho\}$ above the curve $\widetilde{\rho}(\alpha)$ it chooses $k^R < 1-\ell^R$ solving $\Pi^B\left(k^R, \ell^R\right) = 0$ and $\ell^R \geq \widehat{\ell}$ being the solution to (47) so that $r_2^B > 1$.

Below the curve $\widehat{\rho}(\alpha)$, there are two candidate equilibria. One corresponds to $\lambda_1 = 0$,

$\lambda_2 > 0$, $k^R = 1 - \ell^R$ and $\ell^R \geq \widehat{\ell}$ solving $\Pi^B\left(k^R, \ell^R\right) = 0$. The other, instead, corresponds to $\lambda_1 = \lambda_2 = 0$ so that (46) gives $k^R = 1 - \underline{\chi}(\ell)$, which in turn implies that $\theta^* \to \underline{\theta} < \theta^E$, while ℓ^R solves $\Pi^B\left(1 - \underline{\chi}(\ell), \ell\right) = 0$. In the two equilibria, the expression for the social welfare is given by

$$SW_{\underline{\theta}} = \int_0^{\underline{\theta}} \underline{\chi}(\ell)\, d\theta + \int_{\underline{\theta}}^1 R(\theta)(1 - \alpha\ell)\, d\theta$$

when $k^R = 1 - \underline{\chi}(\ell)$ and

$$SW_{\theta^E} = \int_0^{\theta^E} \widehat{\ell} d\theta + \int_{\theta^E}^1 R(\theta)\left(1 - \alpha\widehat{\ell}\right) d\theta$$

when $k^R = 1 - \widehat{\ell}$.

When $\alpha = 0$, the difference between $SW_{\underline{\theta}}$ and SW_{θ^E} is equal to

$$SW_{\underline{\theta}} - SW_{\theta^E} = -\int_0^{\underline{\theta}} \left[\widehat{\ell} - \underline{\chi}(\ell)\right] d\theta - \int_{\underline{\theta}}^{\theta^E} \left[\widehat{\ell} - R(\theta)\right] d\theta < 0,$$

since $\underline{\theta} < \theta^E$ and $R(\theta) < \widehat{\ell}$ for any $\theta < \theta^E$, given the definition of θ^E as given by (16). For any $\alpha > 0$, for $SW_{\underline{\theta}} > SW_{\theta^E}$ to hold α must be sufficiently high, we denote as $\overline{\alpha}$ the threshold value of α for which $SW_{\underline{\theta}} = SW_{\theta^E}$ so that for $\alpha > \overline{\alpha}$ $SW_{\underline{\theta}} > SW_{\theta^E}$ holds. Thus, for $\alpha > \overline{\alpha}$ as long as $\rho < \widehat{\widehat{\rho}}(\alpha)$ so that $\Pi^B\left(1 - \underline{\chi}(\ell), \ell\right) \geq 0$ for $\ell < \widehat{\ell}$, the regulator optimally chooses $k^R = 1 - \underline{\chi}(\ell)$. Given that $\Pi^B\left(1 - \underline{\chi}(\ell), \ell\right)$ is decreasing in both α and ρ, there exists a curve $\underline{\rho}(a)$, with $\underline{\rho}'(a) < 0$ such that in the region below $\underline{\rho}(a)$ and $\alpha > \overline{\alpha}$ the regulator chooses $k^R = 1 - \underline{\chi}(\ell)$. For any other pairs $\{\alpha, \rho\}$ below the curve $\widehat{\rho}(\alpha)$, the regulator chooses, instead, $k^R = 1 - \ell^R$.

Consider now the case when the curve $\widehat{\rho}(\alpha)$ lies above the curve $\widetilde{\rho}(\alpha)$, that is $\widehat{\ell}$ is large. Using the same argument as above, for pairs $\{\alpha, \rho\}$ below $\underline{\rho}(a)$ and $\alpha > \overline{\alpha}$, the regulator chooses $k^R = 1 - \underline{\chi}(\ell)$, otherwise the allocation featuring $k^R = 1 - \ell^R$ and $\ell^R = \widehat{\ell}$ emerges as the equilibrium to the regulator's problem. In the region above $\widetilde{\rho}(\alpha)$, based on the same arguments established above, the equilibrium features $k^R < 1 - \ell^R$ solving $\Pi^B\left(k^R, \ell^R\right) = 0$ and $\ell^R \geq \widehat{\ell}$ being the solution to (47) so that $r_2^B > 1$ and the proposition follows. □